Module 7: Using algebraic reasoning

Description This module is for an individual teacher or group of teachers in secondary schools who are considering their teaching of algebra. It discusses ways of using magic squares and addition grids to help pupils to apply algebraic reasoning and, for pupils working confidently at level 5 or above, to prove their results.

Other activities which could be combined with this one, either to create a longer session, or to work through in a sequence over time are:

- Module 5: Collecting like terms
- Module 10: Classroom approaches to algebra

Study time About 60 minutes

Resources Each teacher will need a personal notepad.

Each teacher or pair of teachers working together will need:

- copies of **Resources 7a, 7b, 7c and 7d** at the end of this module, and scissors;
- a copy of the algebra strand of the *Revised learning objectives for mathematics* for Key Stages 3 and 4 produced by the Secondary National Strategy (2010), which you can download from: nationalstrategies.standards.dcsf.gov.uk/secondary/framework/maths/fwsm/mlo

Using algebraic reasoning

1 When pupils simplify expressions by collecting like terms, the complexity of the expressions can be varied and, where pupils play the game in groups, different sets of cards can be given to different groups.

Other activities which involve collecting like terms offer further mental practice but also require reasoning to solve algebraic problems. An example of this kind of activity is one involving magic squares. In a magic square, a set of numbers is arranged in a square grid so that the numbers in each row, column and the two main diagonals have the same total.

On your notepad, quickly arrange the numbers 1 to 9 to form a 3 by 3 magic square. This problem helps pupils to practise adding three single-digit numbers mentally but, in addition, requires some reasoning in order to arrive at a solution.

A teacher who wants pupils who have solved this problem to describe their methods and reasoning is likely to prompt them with questions such as:

- Where did you start?
- What did you do next? Why?
- How many different solutions are there?

Through discussion, the teacher will help pupils to appreciate that although solutions may look different, they are all reflections and rotations of each other. There is only one solution represented in different ways. The central number is the middle number of the set of numbers 1 to 9 (and is also the mean of the set).

2 The same conclusion can be reached by algebraic reasoning.

Cut out and shuffle up the nine cards, each with an algebraic expression, on **Resource 7a, An algebraic magic square.** Arrange the cards to form a magic square.

When you have completed the square to your satisfaction, consider these questions.

- How did you start? How did you continue?
- Could you have started and continued in a different way?
- How did you check your solution?
- How many different solutions can you find?
- To what extent did your earlier consideration of a magic square with the numbers 1 to 9 help you to solve the problem with algebraic expressions?

Some points to remember in relation to the algebraic magic square activity are these.

- The problem demands logical reasoning and other problem-solving skills.
- The solution can be checked by determining whether the sum of the three algebraic expressions in each line is the same. It can also be checked by substituting particular values for a and b (e.g. $a = 1$, $b = 0$, or $a = 0$, $b = 1$).
- There is one solution. Reflections and rotations produce other possible arrangements of this solution.

$a + 4b$	$8a + 3b$	$3a + 8b$
$6a + 9b$	$4a + 5b$	$2a + b$
$5a + 2b$	$7b$	$7a + 6b$

- Substituting different values for a and b produces an infinite number of different magic squares, all with the same structure, rules and internal relationships as each other.
- This approach is one way of illustrating how the handling of algebraic expressions can grow out of familiarity with handling numbers. Working first with the original 1 to 9 number square helps to provide insights into solving the algebraic square.

3 The next activity involves collecting like terms in the context of addition grids. Once again, reasoning is involved.

Complete the addition squares and answer the questions on **Resource 7b, Addition squares**.

4 Compare your explanations and justifications on **Resource 7b** with those below.

1 *An explanation and justification of the solution to the second of the first pair of problems might go like this.*

For the top left entry:

97 + ☐ = 126

The number in the box must be 29, since 126 – 97 = 29.

For the bottom left entry:

29 + ☐ = 178

This time, the number in the box must be 149, since 178 – 29 = 149 …
… and so on.

2 *An explanation and justification of the solution to the second of the second pair of problems might go like this.*

For the top left entry:

$(3f + 4g) + (\square + \square) = 4f - 3g$

The terms in the boxes must be f and ^-7g, giving the expression $f + {}^-7g$ or $f - 7g$.

For the bottom left entry:

$(f - 7g) + (\square + \square) = 3f - 5g$

This time, the terms in the boxes must be $2f$ and $2g$, giving $2f + 2g$ …
… and so on.

5 Now consider the questions on **Resource 7c, Reflections on addition squares**.

Compare your answers to the questions on **Resource 7c** with those below.

- What do you notice about the diagonals in the addition squares on Resource 7c?
  Each diagonal has the same total.
- Prove that this must always be the case for a 2 by 2 addition square.
  A proof that each diagonal in a 2 by 2 addition square must have the same total can be derived from a square like this.

+	a	b
c	$a + c$	$b + c$
d	$a + d$	$b + d$

The total of the cells on the diagonal from top left to bottom right is
$(a + c) + (b + d) = a + b + c + d$

The total of the cells on the diagonal from top right to bottom left is
$(b + c) + (a + d) = a + b + c + d$

- What are the advantages of moving explicitly from numbers to algebra in activities like these?
  Pupils can draw the parallels between the arithmetic and algebraic processes. Their understanding of what happens with numbers helps them to understand and generalise what happens with algebraic expressions.

6 Complete the addition squares and answer the questions on **Resource 7d, More addition squares**. This time you are given the 'output' expressions and need to think about what the missing 'input' expressions might be.

7 Compare your answers to the questions on **Resource 7d** with those below.

- How many solutions are there to each of the addition square puzzles?
  All except the third example have an infinite number of solutions. The third example has no solution.
- Prove your statements about the number of solutions.
  Proofs about the number of solutions might go like this.

  The third example has no solution, since in a complete 2 by 2 addition square the diagonals must have the same total. In this example, each diagonal has a different total.

In the first, second and fourth puzzles, there is a limitless choice for the first and, therefore, subsequent inputs. For example, put an unrelated term (say z) as an input that can take any value. This produces a workable solution.

+	$3a + 5b - z$	$5a + 3b - z$
z	$3a + 5b$	$5a + 3b$
$z - a + 2b$	$2a + 7b$	$4a + 5b$

4 Some important principles in the teaching of algebra are as follows.

- Pupils need to develop understanding that algebra is a way of generalising. It helps pupils when they are aware that what works with numbers works also with algebraic expressions. There are stimulating activities that can be developed from work with numbers and which allow pupils to practise simplifying or transforming algebraic expressions. Many of these activities also involve algebraic reasoning.
- Pupils should be asked to explain their solutions to algebraic problems and to justify their mathematical reasoning. Pupils working confidently at level 5 or above should be asked to prove their results.

To round off, reflect on these questions.

- What have you learned by participating in these activities?
- What action will you take as a result?

Jot down any points to follow up in further study and any modifications you will make to your planning or teaching.

If you are studying alone, jot down any points you want to discuss with your head of department or other colleagues.

5 You could if you wish follow up this module by reading and using lesson A4 from the series *Improving learning in mathematics: Mostly algebra* (sessions A1–A14). These materials were originally published in 2005 by the Standards Unit at the then Department for Education and Skills (DfES). They can be downloaded from the Learning and Skills Improvement Service (LSIS) Excellence Gateway:

tlp.excellencegateway.org.uk/pdf/mat_imp_02.pdf

Resource 7a: An algebraic magic square

Cut out and shuffle up the nine cards below.

Arrange the cards to form a magic square.

$a + 4b$	$2a + b$	$8a + 3b$
$5a + 2b$	$7a + 6b$	$7b$
$6a + 9b$	$4a + 5b$	$3a + 8b$

Resource 7b :Addition squares

1 Complete these two addition squares.

+	8	14
3	…	…
17	…	…

+	…	…
97	126	…
…	178	537

What number skills are being practised?

What other mathematical skills are involved?

How would you explain and so justify your solution to someone else?

Resource 7b :Addition squares [continued]

2 Complete these two addition squares.

+	$2c + 3d$	$8c + 2d$
$4c + 5d$	…	…
$3c + d$	…	…

+	…	…
$3f + 4g$	$4f - 3g$	…
…	$3f - 5g$	$2f - g$

What algebraic skills are being practised?

What other mathematical skills are involved?

How would you explain and so justify your solution to someone else?

Resource 7c: Reflections on addition squares

What do you notice about the diagonals in the addition squares on Resource 7b?

Prove that this must always be the case for a 2 by 2 addition square.

What are the advantages of moving explicitly from numbers to algebra in activities like these?

Resource 7d: More addition squares

Complete these addition squares.

+	…	…
…	$3a + 5b$	$5a + 3b$
…	$2a + 7b$	$4a + 5b$

+	…	…
…	$2g + 5h$	$6g - 3h$
…	$4h - g$	$3g - 4h$

+	…	…
…	$8t - 3u$	$5t - 4u$
…	$15t - u$	$2t + 6u$

+	…	…
…	$5a^2 + 8ab$	$6a^2 - 3ab$
…	$2a^2 + 10ab$	$3a^2 - ab$

When you have worked on each of the puzzles for a few minutes, make some notes on your answers to the questions.
How many solutions are there to each of the addition square puzzles?

Prove your statements about the number of solutions to each puzzle.

