Enlargement and similarity

Enlargements and similarity are applications of ratio and proportion. Making this link plays an important part in helping pupils to see the 'big picture'. Mental methods that pupils develop to solve ratio and proportion problems can be extended to their work in shape and space. Equally, work in shape and space can provide a context in which pupils are expected to explain and justify the mental approaches that they use. This is crucial if pupils are to extend their understanding of enlargement into trigonometry. The intention is that pupils understand the processes involved and make connections in their own learning rather than struggling to apply a set of 'learned rules' to challenging contextual problems.

Classroom experience suggests that pupils' understanding of enlargements and similarity can be developed through a range of practical and mental activities, for example:

- using centres of enlargement to generate whole-number, fractional and negative enlargements;
- comparing lengths within and between a shape and its enlargement, and distances from the centre of enlargement;
- using ICT and photocopiers to generate enlarged shapes, and discussing scale factors associated with different paper sizes, including the international 'A' series.

Such experiences help pupils to develop their mental imagery as a vital tool in identifying how shapes are enlarged, in recognising their common features and in building strategies that can help them to solve problems involving applications of enlargement and similarity.

Understand and use enlargements and similarity:
recognising and visualising enlargements – positive whole-number scale factors – positive fractional scale factors – negative scale factors
using and interpreting effects of scale factors of enlargements
understanding and applying similarity and congruence
understanding and using Pythagoras' theorem when solving problems in two and three dimensions
beginning to use sine, cosine and tangent in right-angled triangles to solve problems in two dimensions.

The Framework for teaching mathematics: Years 7, 8 and 9, supplement of examples, pages 213 to 215, provides contexts in which pupils should develop mental processes in enlargements and similarity.

Further support is available in the Key Stage 3 National Strategy publications:

- *Key Stage 3, Mathematics study module*, module 8: Ratio and proportion 2; (DfES 0156/2004 G)
- *Interacting with mathematics in Key Stage 3, Year 9, Proportional reasoning minipack*, page 23, 'Cat faces' and 'Photographic enlargements', both supported by an interactive teaching program (ITP) that can be used on a laptop/projector or an interactive whiteboard; (DfES 0588/2002 G)
- *Embedding ICT @ Secondary: Mathematics*, Section 4, Case Study 1. (DfES 0812/2004)

Ray visualisers provide an introduction to enlargements by using prepared enlargements shown on lines radiating from the centre of enlargement. This can be done on paper or using technology (ICT).

On **paper**, prepare large display copies or OHT copies of the resource sheets shown on pages 36 to 38. Show resource 3: *Rays of enlargement A*, and ask pupils to imagine a small triangle with a vertex on each ray. Next, ask them to imagine a triangle with sides that are twice as long as in the original, still with a vertex on each ray. Ask:

- ☐ How far up the rays does the larger triangle sit?

Show resource 4: *Rays of enlargement B*, and take different responses to these questions:

- ☐ Which angles are the same?
- ☐ Which lengths are doubled?

Show resource 5: *Rays of enlargement C*, and ask pupils:

- ☐ Which of the two triangles is the correct enlargement?

They should explain why.

The challenge can be varied by preparing more resource sheets:

- ☐ starting from a larger object shape and enlarging with a scale factor between 0 and 1;
- ☐ using rays that radiate in both directions to allow for a negative scale factor;
- ☐ repositioning the centre at a vertex of the object shape, on a side of the object shape or within the object shape;
- ☐ showing a pair of triangles without rays and asking where the centre could be.

Using **technology** such as dynamic geometry software on an interactive whiteboard (or a computer projection system), create images A, B and C, as on the resource sheets on pages 36 to 38, and proceed as above.

The software can considerably enhance the task by allowing the pupils to think about what happens to the image if the centre of enlargement is moved. Their hypotheses can be tested immediately and subsequently refined.

As pupils drag the centre of enlargement to different locations on the screen, ask the class to speculate about what stays the same and what changes.

By asking appropriate questions, ensure that pupils understand that when the sides of the triangle are doubled in length the distances from the centre of enlargement are also doubled.

Now construct a second image from the original object, using the same centre of enlargement, but this time with a scale factor 3. Ask similar questions for the new shape.

- ☐ Which angles are the same?
- ☐ Which lengths are trebled?

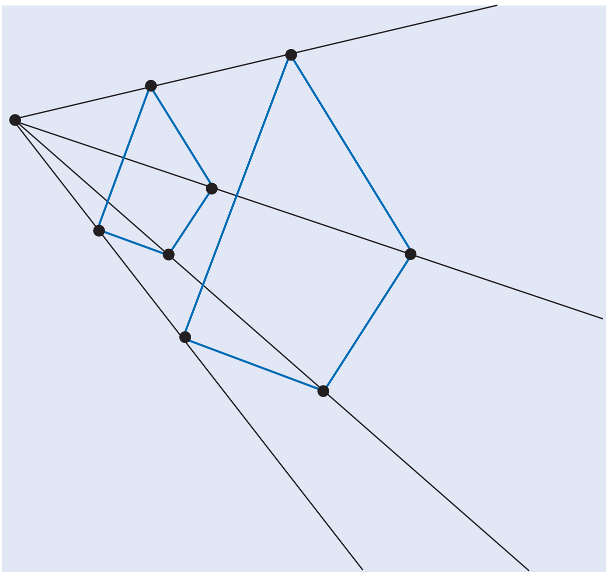
Then ask:

- ☐ What would a scale factor of $\frac{1}{2}$ (or –2) mean? How would you construct the enlargement? Is the image in the same position? Why/why not?
- ☐ What about scale factors of $\frac{1}{3}$, –3, $\frac{2}{3}$, $-\frac{2}{3}$, –0.1 … ?
- ☐ How could you predict their positions?

Take feedback about each of the above scale factors in turn, ensuring that pupils explain and justify their conjectures. Change the scale factor to show the result and discuss the accuracy of their suggestions.

Use this discussion to draw out general principles and applications of enlargements by scale factors greater than 1, between 0 and 1, and less than 0.

The *Framework for teaching mathematics: Years 7, 8 and 9*, supplement of examples, pages 213 and 215, provides examples on applications of enlargements.



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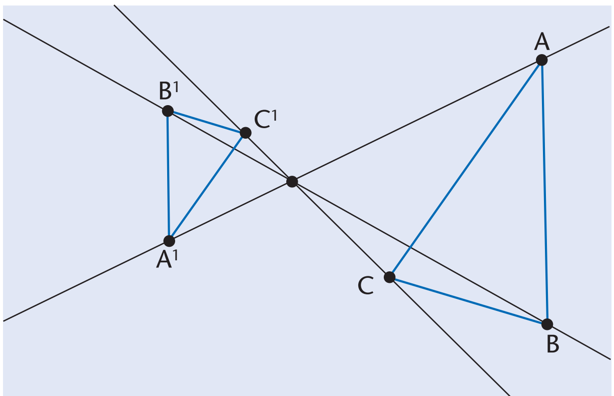
Always, sometimes or never true? can generate interesting discussion. Compile a set of statements and ask pupils to classify them as 'always true', 'sometimes true' or 'never true'. For example:

- If a shape is enlarged by a scale factor 2, then the perimeter of the image is doubled.
- Enlargements produce larger shapes.
- If a shape is enlarged by a scale factor 2, then the area of the image is doubled.

Intervene with pairs who are 'stuck' by asking them to try to visualise or draw one shape that confirms each statement and another that does not. Pupils must use geometrical argument to justify their conclusions. Follow-up prompts might include:

- Is it *necessarily true* that if a shape is enlarged by a scale factor 2, then the perimeter of the image is *always* doubled? Put together a chain of reasoning that will convince another pair.
- Do enlargements *necessarily* produce larger shapes? When does this happen? When does it *not* happen?
- Is it possible for the area of the image to be doubled when a shape is enlarged by a scale factor 2? Explain what happens to the area if the sides are doubled. Under what circumstances does the area double?

When pupils engage in rich tasks their understanding of enlargement and similarity will develop. As misconceptions become apparent, stimulate discussions of the kind described above, perhaps using an 'always, sometimes, never' approach. Further examples on similarity can be found in the *Framework for teaching mathematics: Years 7, 8 and 9*, supplement of examples, page 193.



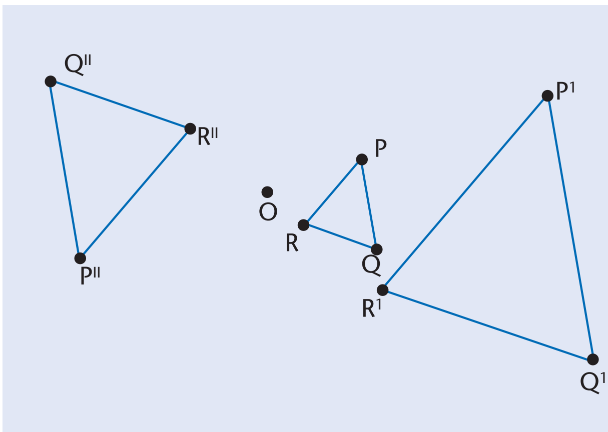
Enlargement by –0.5 scale factor

In **Matching**, pupils make decisions about congruence or, later, similarity of shapes. They are asked to judge, from given information, whether a shape would have the same form, particularly the same angles, no matter how they tried to construct it. Take as an example a parallelogram with sides of 4 cm and 5 cm and one angle of 134°. For most people, this is a mental activity involving checking the orientation and relative positions of angles and sides that correspond to one another. Once pupils have successfully completed the matching process, congruence usually reveals other properties, while similarity tends to lead to work with ratios to establish lengths of sides. Thus a mental matching task could be a precursor to more detailed written checking and justification.

On resource 6: *Enlargement grid*, on page 39, the triangle ABC has been enlarged from different centres of enlargement, using different scale factors, to produce triangles DEF, GHI, JKL, MNO, PQR, STU, VWX, YZA$_1$ and B$_1$C$_1$D$_1$. Ask pupils to locate the centre and scale factor of each enlargement. Tell them to beware because one of the triangles is not an enlargement of the original. Ask pupils to identify which triangle this is and to move one of its vertices so that it is an enlargement of the original.

Once pupils have completed this task, a good plenary activity is to ask pupils to move between the various enlargements. For example, given ×3 and ×–2 enlargements of the same figure, they could be asked:

- What scale factor is involved in moving between those two enlargements?
- Where would the centre be?



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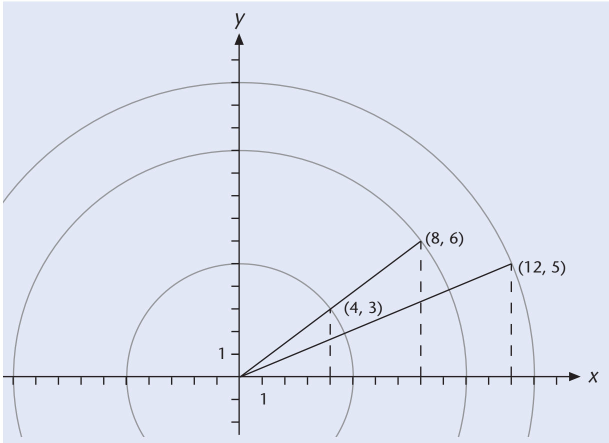
Providing a commentary requires pupils to work on an illustration of Pythagorean triples. Give each pair of pupils a large sheet of 5-mm squared paper showing concentric circles centred on the origin. Choose radii that are multiples of 5, 13, 17 or 25. Ask pupils to mark, on the circles, points that have integer (x, y) coordinates. They should then draw a right-angled triangle with sides connecting the point on the circle to the origin and a point on the axis.

Their task now is to annotate the diagram, which could be pasted in the centre of an A2 sheet of paper. Pairs should use their notes to describe what they notice about the lengths of the sides of each triangle and how the triangles compare to each other.



For example:

- ☐ Are the points accurate? The point (13, 11) does not quite lie on the circle of radius 17. How can you use Pythagoras' theorem to check? (Look at the units digits.)
- ☐ How are some triangles linked by transformations?

If pupils fail to notice similar triangles, suggest that they look at rays from the origin through some sets of points. For example, for circles with radii 5, 10, 15 and 20 units, the ray through (3, 4) also passes through (6, 8), (9, 12) and (12, 16).

The question, 'How many Pythagorean triples exist?' is a straightforward one, since there are infinitely many enlargements of (3, 4, 5), for example. A more difficult question to answer is, 'If (3, 4, 5), (6, 8, 10), … represent one family of Pythagorean triples, how many different *families* of Pythagorean triples are there?'

Earlier applications of Pythagoras' theorem can be found in the *Framework for teaching mathematics: Years 7, 8 and 9*, supplement of examples, page 189.

Photographic enlargement can be used to introduce trigonometry through enlargement. Trigonometry has its roots in similarity. The trigonometric ratios use proportional reasoning to compare any given right-angled triangle with another in which the hypotenuse is of unit length. The idea is developed as an interactive teaching program (ITP) for use on projectors or interactive whiteboards and is available on the web or the CD-ROM as part of the *Interacting with mathematics in Key Stage 3, Year 9, Proportional reasoning minipack* (DfES 0588/2002). Understanding the equivalence of trigonometric ratios for similar triangles hinges on understanding that the ratio of sides within similar shapes is maintained under enlargement.

Use the ITP or show the set of similar triangles copied from those shown on resource 7: *Photographic enlargement*, on page 40. Use the OHT, poster-size copy or A3 sheets on tables. Explain that:

- ☐ This is a set of triangles that are enlargements of one another – it is a set of similar triangles.
- ☐ There are 18 figures missing from the diagram (*on a poster have these values hidden by small sticky notes*), but they cannot be found by measuring as the triangles are not drawn accurately.
- ☐ Of the missing values, six are dimensions (the two shorter sides of the triangle), six are scalings between the three triangles in both directions and six are internal ratios written as scalings from one side to the other and vice versa (these six are the tangents of two acute angles in the triangles).

Set the class this challenge.

I will give you some of the values. You have to ask for the minimum you need to work out all the other values. What values do you want me to give? (Which flaps will you choose to lift?)

Use the task to focus on the fact that the internal ratios are the same for all the triangles regardless of the scalings between them.

Discussion should also use the fact that knowing a side and an internal ratio means that you do not need to reveal the other side because it can be calculated.

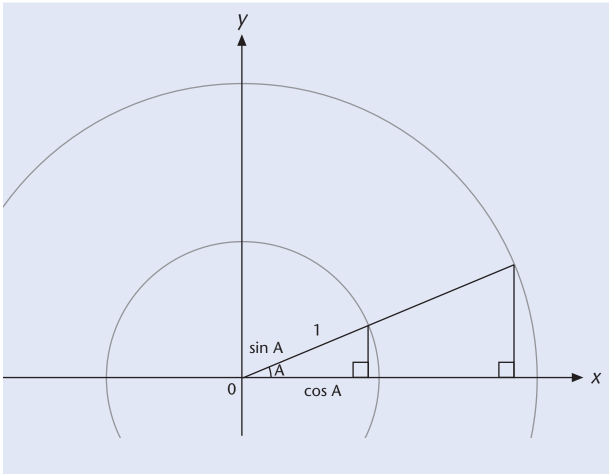
The *Framework for teaching mathematics: Years 7, 8 and 9*, supplement of examples, pages 191 and 193, includes examples on congruence and similarity. These are supported by interactive software available from the website of the Mathematical Association (**www.m-a.org.uk**).

Triangles on a unit circle can be used as a follow-up to photographic enlargement and as a precursor to work on the rotating unit arm to generate trigonometric curves. Effectively, the pupils are asked to annotate nested similar triangles based on the connections they have formed from working with the photographic enlargement task.

Give each pair of pupils a large sheet of 5-mm squared paper showing concentric circles centred on the origin. On these diagrams pupils should draw some sets of similar right-angled triangles with a vertex at the origin, as shown below. The diagrams could be pasted in the centre of an A2 sheet of paper for pairs to add annotations describing what they know about:

- how the triangles compare to each other;
- the ratio of the lengths of pairs of sides within a set of triangles.

The concluding discussion should establish that ratios in the triangles on the unit circle can be generalised, through enlargement, to every right-angled triangle.



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Sorting right-angled triangles is a matching activity that supports pupils' reasoning about, and facility with, trigonometry. It is an end-of-unit or revision activity, rather than an introduction to the topic. The pupils should not work with pencil and paper or a calculator. The aim of the task is to increase the facility with which pupils connect the dimensions shown on the triangle with the trigonometric ratio.

In the activity, pupils are provided with cut-out copies of cards produced from resource 8: *Right-angled triangles for sorting*, on page 41, and resource 9: *Trigonometric ratios for matching*, on page 42. Working in pairs or small groups, they sort the triangles according to whether they are similar and match the trigonometric ratios for the angle marked in each triangle. In the triangles, some of the ratios repeat, so the numbers of cards in the two sets are not equal. Please note the additional challenge involved in sorting some cards, which involves simplification of surds, such as $\sqrt{45} = 3\sqrt{5}$.

To increase the challenge further, pupils could be asked to reduce the number of matched sets to a minimum by merging equivalent sets. This involves mental use of Pythagoras' theorem, for example, the set for $\cos x = \frac{1}{\sqrt{2}}$ is equivalent to the set for $\tan x = 1$.

The *Framework for teaching mathematics: Years 7, 8 and 9*, supplement of examples, pages 191 and 193, contains examples on congruence and similarity. These are supported by interactive software available from the website of the Mathematical Association (**www.m-a.org.uk**).