Interacting with mathematics in Key Stage 3

Year 8 multiplicative relationships: mini-pack

Contents

Year 8 multiplicative relationships: sample unit

Introduction

This Year 8 unit has been developed through a flexible use of the *Sample medium-term plans for mathematics.*

In planning the unit several decisions were made that affect the medium-term plans for mathematics.

- The multiplicative relationships unit is additional to those in the *Sample medium-term plans.*
- The objectives for the unit are from the strand on number, so the list of objectives addressed in Number 2, Number 3 and Number 4 is reduced.
- It is taught during the spring term of Year 8.
- Number 2 is covered in the autumn term.
- Number 3 and Number 4 are taught after this unit.
- The objectives for the unit Solving problems are addressed in the sequence Number 2, Multiplicative relationships (this unit), Number 3 and Number 4.

Understanding proportionality provides the key to much of the Key Stage 3 mathematics curriculum. In number, proportionality occurs in work with fractions, decimals, percentages, ratio and rates; in algebra, proportions ($y = mx$) are a subset of linear functions ($y = mx + c$); in geometry, proportionality occurs in ideas of scale and enlargement; in handling data, it underpins many statistical measures, graphical representations and probability. Proportional thinking is essential in the solution of many problems. This may not be immediately apparent and may not be accessible through a single technique such as the unitary method. The underlying ideas must therefore be taught systematically.

This unit has been structured into three phases for teaching.

Phase 1 (about four lessons)

- Addresses the need to move from additive to multiplicative thinking, introducing the idea of scaling numbers (including multiplicative inverses) and identifying proportional sets.
- Works with ‘difficult’ numbers (i.e. numbers that have a remainder when divided), forcing attention to the mathematical relationships and generalities involved.
- Emphasises fraction, decimal and percentage equivalents, treating the latter as a special kind of decimal (hundredths).

Phase 2 (one lesson)

- Involves identifying practical examples of proportions, leaving further extension to link with work on linear functions.

Phase 3 (about four lessons)

- Examines problems, mainly within the field of number, which are solved by multiplication or division, or a combination of the two operations.

- Highlights strategic approaches to problems by systematically examining the stages of a solution. These will include extracting the data, clarifying the relationships involved, identifying what operations are needed, and considering the meaning and likely size of numbers at each stage of the solution.
- Emphasises the use of calculators, seeking to automate the process of calculation, using the operational understanding developed in the first phase of the unit.

Objectives

A Understand multiplication and division of integers and decimals; use the laws of arithmetic and inverse operations; check a result by considering whether it is of the right order of magnitude.

B Use division to convert a fraction to a decimal; calculate fractions of quantities; multiply (and divide) an integer by a fraction.

C Interpret percentage as the operator 'so many hundredths of' and express one given number as a percentage of another; **use the equivalence of fractions, decimals and percentages to compare proportions; calculate percentages and find the outcome of a given percentage increase or decrease.**

D Consolidate understanding of the relationship between ratio and proportion; reduce a ratio to its simplest form, including a ratio expressed in different units, recognising links with fraction notation; **divide a quantity into two or more parts in a given ratio; use the unitary method to solve simple word problems involving ratio and direct proportion.**

E Identify the necessary information to solve a problem, using the correct notation and appropriate diagrams.

F Solve more complex problems by breaking them into smaller steps, choosing and using efficient techniques for calculation.

G Suggest extensions to problems, conjecture and generalise; identify exceptional cases or counter-examples.

Differentiation

- The unit references the Framework's supplement of examples. For each objective the pitch of the work is accessible through Years 7 to 9 and examples can be chosen appropriately.
- Ideas and strategies for progression in oral and mental starters are provided in a set of prompts (see pages 10–15).
- Phase 3 is supported by a bank of Key Stage 3 test questions ranging from level 5 to level 7 (see pages 23–28).
- This is only the formative stage of securing understanding of proportional relationships. Pupils at all levels should have their thinking challenged by the 'big ideas' introduced throughout the unit, and all will need further work to clarify their thinking and consolidate their skills

Resources

- Calculators
- Pupil resource sheets (included in the school file):
  - Parallel number lines: examples
  - Parallel number lines
  - Multiplicative relationships: key results
- Supplementary notes (pages 10–28 of the mini-pack):
  - Prompts for oral and mental starters
  - Prompts for main activities in phase 1
  - Prompts for main activities in phase 3
  - Problem bank for phase 3

Key mathematical terms and notation

scale factor, multiplier, operator

inverse operation, inverse operator, multiplicative inverse

ratio (including notation $a : b$), fraction, decimal fraction, percentage (%)

proportion, direct proportion

rate, per, for every, in every

unitary method

Unit plan

Oral and mental starter

Objectives B, D
(See prompts for language and visual images to use; also Framework p. 69)

Say fraction tables aloud (whole class together or taking turns), in different forms:
$1 \times \frac{1}{4} = \frac{1}{4}$, $2 \times \frac{1}{4} = \frac{1}{2}$, …

$\frac{1}{4}$ of $1 = \frac{1}{4}$, $\frac{1}{4}$ of $2 = \frac{1}{2}$, …
How do you find $\frac{1}{4}$ of a number?
(Link 'of' with ×.)

$\frac{1}{4}$ of $6 = \frac{6}{4} = 1\frac{1}{2}$, $\frac{2}{4}$ of $6 = 3$, $\frac{3}{4}$ of $6 = 4\frac{1}{2}$, $\frac{4}{4}$ of $6 = 6$, …
How do you find $\frac{5}{4}$ of a number?
(Relate to $\div 4 \times 5$ and to $\times \frac{5}{4}$.)

Generate proportional sets in an ordered sequence: e.g. multiples of 10 with multiples of 4, alternating between terms of the two sets.

Main teaching

Phase 1 (four lessons)
Objectives A, B, C, D, G (Framework, pp 65, 67, 71)
Scaling numbers (perhaps two lessons)
How can you get from 5 to 8 using only multiplication and division?
$5 \div 5 \times 8$, $5 \times 8 \div 5$ or $5 \times \frac{8}{5}$ ($\frac{8}{5}$ of 5).
Inverting this, how can you scale from 8 to 5? $8 \div 8 \times 5$, $8 \times 5 \div 8$ or $8 \times \frac{5}{8}$ ($\frac{5}{8}$ of 8).
Illustrate different methods graphically with line segments on sets of parallel number lines.
From similar examples establish $a \times \frac{b}{a} = b$, also dividing by a then multiplying by b is equivalent to single operation of multiplying by $\frac{b}{a}$ (called a multiplier or scale factor).
Establish (multiplicative) inverse.

Consider decimal and percentage forms of scale factors: $\times \frac{8}{5} = \times 1.6 = 160\%$ and the inverse $\times \frac{5}{8} = \times 0.625 = \times 62.5\%$ (and probably $\div 1.6$ as well).

Ratio and proportion
Consider relationships between two sets of numbers a and b. Identify multipliers for each pair of entries. Multiplier can also be called the ratio $b : a$. Where ratio is equivalent for each pair of numbers, the sets of numbers are in proportion. Establish inverse ratio $a : b$.

Give more tables of numbers: identify which sets of numbers are in proportion and which are not (for those in proportion, identify ratios $a : b$ and $b : a$).
If time allows, draw graphs of proportions, noting enlarging triangles.

Using ratio and proportion
Given sets of numbers in proportion, identify appropriate ratios and use to calculate unknown entries:

a	b
3	5
7	x
9	15
10.5	17.5

$x = 7 \times \frac{5}{3}$

a	3	y	9	10.5
b	5	20	15	17.5

$y = 20 \times \frac{3}{5}$

Sets of numbers reduced to two entries can now be thought of as rows or columns – with unknown in any of the four positions:

a	b
3	5
x	6.4

$x = 6.4 \times \frac{3}{5}$

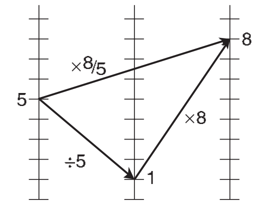
or

a	3	5
b	x	6.4

$x = 3 \times \frac{6.4}{5}$

Notes

Use pupil resource sheet 'Parallel number lines':

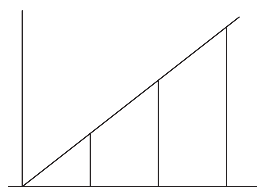


Support: Start with easier numbers and scale factors. Introduce idea of multiplicative inverse but spend less time on it.

All pupils cover fraction, decimal and percentage forms, treating percentages as hundredths.

Extension: Pupils to make up tables for a partner to explore.

Link to shape and space work on enlargement (Shape, space and measures 3; Framework, pp. 212–5):



Support**:** Keep unknown in second column. Start with easy scale factors.

Extension**:** Give examples with more than one unknown. Ask pupils to consider more than one way of finding an entry.

Plenary

Ask pupils to demonstrate examples and clarify methods.

Link to oral and mental starters:

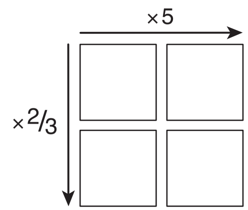
- $\frac{3}{4}$ and $4\frac{1}{2}$ are in the same fraction table, what could it be?
- (20,16) is a corresponding pair in proportional sets, what might the two previous pairs be?

Deal with issues relating to fraction, decimal, percentage conversion.

Ask pupils if they can generalise results, particularly that scale factor from a to b is $\frac{b}{a}$ and from b to a is $\frac{a}{b}$.

Given scale factor, pupils use calculators to generate tables of numbers in proportion. Whole class to check some values.

Related idea:



What numbers could go in the boxes? Is there a unique set?

Hand out pupil resource sheet 'Multiplicative relationships: key results'. Work through it, asking pupils to give examples of their own.

<table>
<tr><th>Oral and mental starter</th><th>Main teaching</th><th>Notes</th><th>Plenary</th></tr>
<tr>
<td>Objective D
Generate proportional sets (as a reminder of what they are).</td>
<td>Phase 2 (one lesson)
Objective D
Describe practical situations and ask class whether they are in proportion. Why? Why not?
Cover these points:
• Suggest units in which quantities might be measured.
• Discuss concept of ‘rate’, linking units by ‘per’ or symbol /.
• Tabulate possible sets of values and, if time, draw graphs.</td>
<td>Types of example:
Distance/time at constant speed
• Weight/cost at given unit price
• Height/weight of a group of people
• Mass attached / stretch in elastic
• Amount of meat / size of burger
Link to algebra of linear functions (Algebra 3 and Algebra 5; Framework pp. 164–7,172–7).</td>
<td>Ask for one or two examples identified as proportions and discuss circumstances under which they might not be so: e.g. travel at varying speed, exchanges of currency made on different days, etc.</td>
</tr>
<tr>
<td>Objectives B, C
(Extend up to 15 minutes. See ‘Prompts for oral and mental starters’; also Framework pp. 61, 65, 73)
Working towards fluency:
• Using calculators with ‘awkward’ numbers
• Rapid conversion between ratio, fraction, decimal and percentage forms
• Numbers and quantities, using rates, clearly stated … per …

Cover these calculations:
• Expressing proportions: a/b
• Comparing proportions: a/b < = > c/d
• Finding proportions: a/b of …
• Comparing quantities: a/b of … < = > c/d of …
• Using and applying rates: e.g. 4 machines need 17 hours maintenance, how long for 7 machines? (‘hours per machine’, multiplier 17/4)
(See ‘Prompts for oral and mental starters’.)</td>
<td>Phase 3 (four lessons)
Objectives C, D, E, F, G (Framework pp. 3, 5, 61, 71–83)
Strategies for solving problems involving multiplication, division, ratio and proportion
Draw on problem bank, including some from shape and space and handling data.

Activities (may include mini-plenaries):
• Choose one problem: discuss alternative strategies for solving; change numbers (e.g. make them more difficult) and consider how methods can be adapted; ask different or supplementary questions from same context. (See ‘Prompts for main activities in phase 3’)
• Choose small set of problems: concentrate on extracting and organising data (e.g. putting into tabular form) before deciding on possible methods of solution, rather than working problems through to an answer. (See ‘Prompts for main activities in Phase 3’)
• Ask pupils to make up similar problems for a partner to solve.
• Give part solutions and ask pupils to continue and complete solution or give a complete solution and ask pupils to evaluate efficiency of strategy chosen and to identify errors.

Problem-solving strategies to be taught:
• Translate problem into a form that helps with the solution: e.g. extract appropriate data and put in tabular form
• Estimate answer: ask ‘Will it be bigger or smaller?’, ‘Will it be greater or less than 1?’, etc.; use knowledge of effect of multiplying or dividing by numbers greater than or less than 1.
• Consider scaling methods by finding a multiplier.
• When using unitary method, involving division, clarify rates expressed part way to a solution: e.g. is it euros per pound or pounds per euro?</td>
<td>Select suitable problems, ranging from level 5 to level 7.

Support: Include problems with ‘convenient’ or easy numbers, making informal/mental methods appropriate.

Extension: Replace with ‘awkward’ numbers, to force attention on general methods or to make problem more difficult.</td>
<td>(Ideas for short plenaries, to be used as appropriate.)
Pupils to specify units on calculated rates:
<table><tr><td>£</td><td>$</td><td>rate</td></tr><tr><td>10</td><td>14</td><td>1.4 __ per __</td></tr><tr><td>10</td><td>14</td><td>0.71 __ per __</td></tr></table>
Pupils to suggest uses of strategies for solving multiplication and division problems in other areas of curriculum. (Possible homework to collect examples from other subject areas.)

Show figures translated out of a proportion problem. Pupils to suggest possible problems:
<table><tr><td>Height</td><td>Width</td></tr><tr><td>3.2</td><td>1.8</td></tr><tr><td>4.8</td><td>?</td></tr><tr><td>?</td><td>0.25</td></tr></table>
Show complete but incorrect solution. Pupils identify existence of error by estimation, nature of error by examination of strategy.</td>
</tr>
</table>

Supplementary notes

Prompts for oral and mental starters

Phase 1

Chant fraction tables in different forms, whole class together or taking turns. For visual support, write equations on the board, or reproduce extended area diagrams of the form described in the booklet *What is a fraction?* (page 6).*

Multiples of a fraction

$1 \times {}^{1}/_{4} = {}^{1}/_{4}$, $2 \times {}^{1}/_{4} = {}^{1}/_{2}$, . . .

Language to use: 'One multiplied by a quarter is a quarter, two multiplied by a quarter is a half, three multiplied by a quarter is three quarters, four multiplied by a quarter is one, five multiplied by a quarter is one and a quarter, . . .'

Or: 'One quarter is a quarter, two quarters are a half, three quarters are three quarters, four quarters are one, five quarters are one and a quarter, . . .'

Count in multiples of other fractions, for example thirds or fifths.

Same fraction of different numbers

${}^{1}/_{4}$ of $1 = {}^{1}/_{4}$, ${}^{1}/_{4}$ of $2 = {}^{1}/_{2}$, . . .

Language to use: 'One quarter of one is a quarter, a quarter of two is a half, a quarter of three is three quarters, a quarter of four is one, a quarter of five is one and a quarter, . . .'

Key question: How do you find ${}^{1}/_{4}$ of a number?

Find other fractions of the sequence of integers, for example thirds or fifths.

Different fractions of the same number

${}^{1}/_{4}$ of $6 = {}^{6}/_{4} = 1{}^{1}/_{2}$, ${}^{2}/_{4}$ of $6 = 3$, ${}^{3}/_{4}$ of $6 = 4{}^{1}/_{2}$, . . .

Language to use: 'One quarter of six is one and a half, two quarters of six is three, three quarters of six is four and a half, four quarters of six is six, five quarters of six is seven and a half, . . .'

Key questions: How do you find ${}^{1}/_{4}$ of a number? ${}^{5}/_{4}$ of a number? ${}^{9}/_{4}$ of a number?

Find quarters of another number.

Start with a different unitary fraction of a number, e.g. thirds or fifths.

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*The booklet *What is a fraction?* was provided to teachers attending the additional support courses *Planning and teaching mathematics* and *Leading developments in mathematics*.

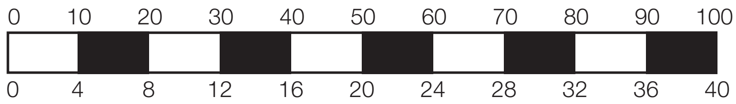
Phase 2

Generating proportional sets

Generate two sets of multiples simultaneously, alternating between the terms of the two sets, for example multiples of 10 and multiples of 4.[†]

Point to markers on the top and bottom of a counting stick; top marker references multiples of 10, bottom marker references multiples of 4. Say the numbers aloud as a class:

> Ten, four, twenty, eight, thirty, twelve, forty, sixteen, . . .



Extend beyond the limit of the counting stick.

Check equality of ratios between the pairs of numbers (use a calculator if appropriate).

Repeat with other pairs of multiples.

[†] See the video of Walt's Y7 lesson on ratio and proportion. The video was included in the school materials (purple box) for the Key Stage 3 conference on the National Numeracy Strategy in the summer term 2000.

Phase 3

Expressing proportions

- **Expect rapid conversion between fractions in lowest terms, decimals, percentages**
- **Expect rates, clearly stated as … per …**
- **Expect use of calculator, when numbers are 'awkward'**

Numbers, first smaller than second

- What is 5 as a fraction of 85?
- What is 5 as a percentage of 85?
- What is 5 as a proportion of 85?

Numbers, first larger than second

- What is 15 as a fraction of 8?
- What is 15 as a percentage of 8?
- What is 15 as a proportion of 8?

Quantities, first smaller than second (Answer is not in units, e.g. £.)

- What is £3 as a fraction of £17?
- What is £3 as a percentage of £17?
- What is £3 as a proportion of £17?

Quantities, first smaller than second, units mixed (Answer is not in units, e.g. £ or p.)

- What is 60p as a fraction of £2?
- What is 60p as a percentage of £2?
- What is 60p as a proportion of £3?

Quantities, first larger than second (Answer is not in units, e.g. kg.)

- What is 23 kg as a fraction of 8 kg?
- What is 23 kg as a percentage of 8 kg?
- What is 23 kg as a proportion of 8 kg?

Reversing proportions (Deal with various forms – 'fractions/decimals/percentages' and 'numbers/quantities'.)

- What is 6 as a proportion of 25?
- What is 25 as a proportion of 6?

Rates to be written as 'this' per 'that'

- 367 miles travelled at a constant speed for 4 hours
- £87 split equally among 6 children
- 14 hours of work to be covered by 9 clerks

Reversing rates (What is the meaning of ‘that’ per ‘this’? When is it useful?)

- hours per mile
- children per pound
- clerks per hour

Comparing proportions

- **Expect rapid conversion between fractions in lowest terms, decimals, percentages**
- **Expect rates, clearly stated as … per …**
- **Expect use of calculator, when numbers are ‘awkward’**

Numbers, first smaller than second

- What is greater: 5 as a proportion of 85 or 7 as a proportion of 90?

Numbers, first larger than second

- What is greater: 26 as a proportion of 8 or 32 as a proportion of 9?

Quantities, first smaller than second (Answer is not in units, e.g. minutes.)

- What is greater: 5 minutes as a proportion of 18 minutes or 3 minutes as a proportion of 10 minutes?

Quantities, first smaller than second, units mixed (Answer is not in units, e.g. £ or p.)

- What is greater: 60p as a proportion of £2 or 90p as a proportion of £2.47?

Quantities, first larger than second (Answer is not in units, e.g. cm.)

- What is greater: 7 cm as a proportion of 18 cm or 6 cm as a proportion of 16 cm?

Reversing proportions (Deal with various forms – ‘fractions/decimals/percentages’ and ‘numbers/quantities’.)

- What is greater: 9 as a proportion of 28 or 13 as a proportion of 40?
- Given the above, what is greater: 28 as a proportion of 9 or 40 as a proportion of 13?

Rates to be written as ‘this’ per ‘that’

- What is the greater rate: 367 miles travelled in 4 hours at a constant speed or 640 miles travelled in 7 hours at a constant speed?

Finding proportions

- **Expect rapid conversion between fractions in lowest terms, decimals, percentages**
- **Expect rates, clearly stated as … per …**
- **Expect use of calculator, when numbers are ‘awkward’**

Numbers, proportions less than 1, some terminating decimals and some not

- What is three fifths of 83?
- What is 0.6 of 83?
- What is 60% of 83?

Numbers, proportions greater than 1, some terminating decimals and some not

- What is ten sixths of 74?

Quantities, proportions less than 1, some terminating decimals and some not (Answer is in units, e.g. £.)

- What is three sevenths of £17?

Quantities, proportions greater than 1, some terminating decimals and some not (Answer is in units, e.g. litres.)

- What is two thirds of 14 litres?

Comparing quantities

- **Expect rapid conversion between fractions in lowest terms, decimals, percentages**
- **Expect rates, clearly stated as … per …**
- **Expect use of calculator, when numbers are 'awkward'**

Numbers, proportions less than 1, some terminating decimals and some not

- Which is greater: one fifth of 83 or one seventh of 90?

Numbers, proportions greater than 1, some terminating decimals and some not

- Which is greater: five thirds of 35 or seven quarters of 33?

Quantities, proportions less than 1, some terminating decimals and some not (Answer is in units, e.g. £.)

- Which is greater: three sevenths of £17 or two fifths of £14?

Quantities, proportions greater than 1, some terminating decimals and some not (Answer is in units, e.g. litres.)

- Which is greater: ten thirds of 14 litres or ten sevenths of 20 litres?

Using and applying rates

- **Expect rapid conversion between fractions in lowest terms, decimals, percentages**
- **Expect rates, clearly stated as … per …**
- **Expect use of calculator, when numbers are 'awkward'**

What is the rate we use in this calculation? (Expect, not a numerical answer, but a rate – e.g. 'hours per machine'.)

- 4 machines need 17 hours of maintenance. How many hours for 7 machines?

What is the rate we use in this calculation? (Expect, not a numerical answer, but a rate – e.g. 'machines per hour'.)

- 4 machines need 17 hours of maintenance. How many machines in 5 hours?

What is the calculation? (Expect, not a numerical answer, but a calculation – e.g. $^{17}/_{4} \times 7$.)

- 4 machines need 17 hours of maintenance. How many hours for 7 machines?

What is the calculation? (Expect, not a numerical answer, but a calculation – e.g. $^{4}/_{17} \times 5$.)

- 4 machines need 17 hours of maintenance. How many machines in 5 hours?

What is the answer?

- 4 machines need 17 hours of maintenance. How many hours for 7 machines?

What is the answer?

- 4 machines need 17 hours of maintenance. How many machines in 5 hours?

Prompts for main activities in phase 1

This sequence of work is planned to be challenging for many pupils. It sets high expectations in order to build on and take advantage of the National Numeracy Strategy in primary schools, which has aimed to improve pupils' fluency in mental calculation. Pupils should come into secondary schools not only with better knowledge of basic multiplication and division facts, but also improved understanding of the relationships between them.

For example, pupils should see

$3 \times 6 = 18$, $6 \times 3 = 18$, $18 \div 6 = 3$, $18 \div 3 = 6$

as a related set of facts – if one is known then the others follow. So, when asked 'What does 3 have to be multiplied by to give 18?' ($3 \times ? = 18$), as well as knowing the answer (6) they should recognise its relationship to the given numbers ($6 = 18 \div 3$). Pupils who have this level of understanding should be ready for the work here.

General teaching points:

- Use calculators to pursue multiplicative relationships between numbers that would be difficult to deal with mentally. This forces pupils to attend to general strategies for dealing with numbers.
- Support pupils, where necessary, by discussing examples with easier numbers, so that they can make connections with what they already know and understand from mental work. They should then return to using calculators with more difficult numbers.

Scaling numbers

How can you get from the first number (referred to as *a*) to the second number (referred to as *b*) using only multiplication?

2	→	5
4	→	5
5	→	8
16	→	5
2	→	3
4	→	3
6.4	→	22.4

Pupils might start with *a* and *b* as whole numbers, but with the ratio $^b/_a$ not a whole number.

Teaching points:

- Line segments drawn on sets of three parallel lines (see the pupil resource sheet 'Parallel number lines') give a strong visual image to support understanding of how two operations can be combined into one: $\div a \times b$ is replaced by the single operator $\times\, ^b/_a$, $^b/_a$ being regarded as a single fraction. It is useful to note that *a* and *b* will not always be integers.
- By reversing the direction of the scalings, pupils can be led to see that $\times\, ^a/_b$ is the inverse of $\times\, ^b/_a$ – treat it as a straightforward matter, not a complicated new idea! This is likely to be the first time they have met the concept of a multiplicative inverse: preserving the operation (multiplication) and seeking an inverse operator ($^a/_b$). The more familiar way, particularly if $^b/_a$ is expressed as a single decimal (e.g. 0.8), is to preserve the operator 0.8 and invert the operation (multiplication) – that is, 'multiply by 0.8' is inverted to 'divide by 0.8'. A possible approach, so as not to introduce too

many complications, is to discuss divisors only where the multiplier is in tenths. The idea of a multiplicative inverse is central to phase 1 of the unit and should be given due emphasis.

- Fractions can be converted easily to decimals ($^b/_a = b \div a$) and percentages (treat percentages as hundredths – of which you can have more than one hundred!). Pupils should have met these conversions before and they need to become fluent with them. Retaining 'of' for × can aid understanding when expressing fractions or percentages of numbers.
- Your department will need to agree on how to deal with recurring decimals – for example, round to 2 decimal places.

There is a lot to cover here, so this material could easily make two one-hour lessons.

Ratio and proportion

a	*b*
7	10.5
0.8	1.2
$1^1/_2$	$2^1/_4$
24	36
1.4	2.1

For all pairs of values, $^b/_a = 1.5$ and $^a/_b = 0.67$ (to 2 decimal places)

Teaching points:

- Look at both examples and counter-examples, such as those considered earlier, to establish the concept of proportion as an equality of ratios. Pupils meet lots of tables of numbers that are well ordered. Avoiding such order here keeps attention on the essential idea behind the concept.
- The terms 'ratio' and 'proportion' are often used together, and their meanings are not always clearly distinguished. For example, in everyday language it is common to refer to a part of a whole as 'a proportion'. It is perhaps wise not to dwell on these distinctions but to allow the context to take care of the meaning. In fact, many questions of a practical nature do not use either term explicitly.
- Pupils need to be familiar with ratio notation $b : a$ (read as 'the ratio of b to a') and know that it is equivalent to $^b/_a$.
- Some graphical work is worthwhile, making a link to the algebra of linear functions and to ideas of enlargement. Time may be short in this unit, however, and it may be better to draw out the links in the next appropriate unit.

Using ratio and proportion

Look at this table of numbers in proportion. What must x be?

6	9
3	x

Using simple numbers can encourage different ways of looking at a problem. In this case, x can be found either by halving 9 or by multiplying 3 by $1^1/_2$. In practical problems the numbers are often simple enough to be dealt with mentally – for example, a recipe for four can be adapted for six by multiplying the amounts by $1^1/_2$. However, as suggested in the unit plan, this phase is concerned purely with relationships between sets of numbers. Practical contexts are left until phases 2 and 3.

Teaching points:

- Pupils need to understand that the unknown number is found by applying a scale factor to one of the known numbers, the scale factor being obtained from a known pair.
- Asking the question 'Will the answer be bigger or smaller?' is a useful check on whether the correct scale factor is being applied.
- This abstract work can be simplified for some pupils by keeping the unknown in one standard position – for example, always the bottom right-hand entry in the table. This is a better strategy than merely simplifying the numbers as it forces pupils to use the ideas of ratio and proportion rather than their knowledge of numbers.

Prompts for main activities in phase 3

Choose one problem: discuss alternative strategies for solving the problem; change the numbers in the problem (e.g. make them more difficult) and consider how the methods can be adapted; ask different or supplementary questions from the same context.

You can make different colours of paint by mixing red, blue and yellow in different proportions.

For example, you can make green by mixing 1 part blue to 1 part yellow.

(a) To make purple, you mix 3 parts red to 7 parts blue.
How much of each colour do you need to make 20 litres of purple paint?
Give your answer in litres.
. . . . . litres of red and . . . . . litres of blue

(b) To make orange, you mix 13 parts yellow to 7 parts red.
How much of each colour do you need to make 10 litres of orange paint?
Give your answer in litres.
. . . . . litres of yellow and . . . . . litres of red

From 1998 Key Stage 3 Paper 2 question 10

Consider possible strategies for part (a). A similar approach can be taken to part (b).

Strategy 1: Mental scaling method

3 litres of red paint plus 7 litres of blue paint makes 10 litres of purple.

Double up for 20 litres.

For which kinds of numbers does this strategy lend itself?

This is the most likely method to come from pupils.

red	blue	purple
3	7	10

Put this table on the board and have the pupils complete the entries.

How does this help us to calculate for 5 litres of purple?

How does this help us to calculate for 13 litres of purple?

How does this help us to calculate for 3.85 litres of purple?

How does it help if the components of red and blue change – for example, 3 parts red and 8 parts blue?

Could I use this strategy to see quickly the percentage of any mix that is made up of red paint?

Other strategies may start to emerge.

Strategy 2: Unitary method

Initial stage to calculate what is required for one unit of the mix.

purple	red	blue
10	3	7
1	3 ÷ 10	7 ÷ 10
20	3 ÷ 10 × 20	7 ÷ 10 × 20

Ask: What calculation takes me from 10 purple to 1 purple?

How does this help us to calculate for 5 litres of purple?

How does this help us to calculate for 13 litres of purple?

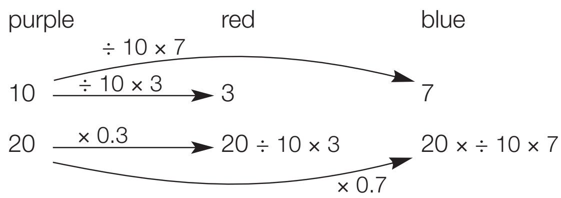
How does this help us to calculate for 3.85 litres of purple?

How does it help if the components of red and blue change – for example, 3 parts red and 8 parts blue?

Could I use this strategy to see quickly the percentage of any mix that is made up of red paint?

Strategy 3: Scale factor method

Finding the proportion of the mix made up by each component part – that is, the factor by which the total is multiplied to calculate each part.



Ask: What calculation takes me from 10 purple to 3 red? What is this as a single multiplier?

How does this help us to calculate for 5 litres of purple?

How does this help us to calculate for 13 litres of purple?

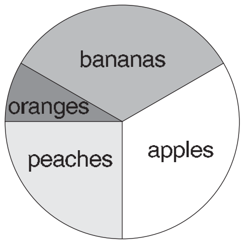
How does this help us to calculate for 3.85 litres of purple?

How does it help if the components of red and blue change – for example, 3 parts red and 8 parts blue?

Could I use this strategy to see quickly the percentage of any mix that is made up of red paint?

Choose a small set of problems: concentrate on extracting and organising the data (e.g. putting into tabular form) before deciding on possible methods of solution, rather than working the problems through to an answer.

- There is 20% orange juice in every litre of a fruit drink. How much orange juice is there in 2.5 litres of fruit drink? How much fruit drink can be made from 1 litre of orange juice?
- This chart shows the income that a market stall-holder got last week from selling different kinds of fruit.



The stall-holder got £350 from selling bananas. Estimate how much she got from selling oranges.

- 6 out of every 300 paper clips produced by a machine are rejected. What is this as a percentage?
- Rena put £150 in her savings account. After one year, her interest was £12. John put £110 in his savings account. After one year, his interest was £12. Who had the better rate of interest, Rena or John? Explain you answer.

From page 75 of the Supplement of examples in the *Framework for teaching mathematics: Years 7, 8 and 9*

Translate the data from each question into a useful form.

Orange juice

20% or 0.2	every 1 litre
20% or 0.2	2.5 litres

What calculation do we perform to find 20% of anything? Roughly how big will the answer be?

Market stall

Bananas	£350	$^1/_3$
Apples		$^1/_3$
Peaches		$^1/_4$
Oranges	?	?

Will the answer be more than £350? Do we need oranges as a fraction of the whole or as a fraction of bananas?

Paper clips

Produced	300	100
Rejected	6	?

What relationship is it useful to identify here?

300 ÷ 3 = 100

Savings

Rena	£150	£12
John	£110	£12

Draw out gut reaction justified with key words.

(a) The label on yoghurt A shows this information.

How many grams of protein does 100 g of yoghurt provide?

Show your working.

Yoghurt A **125 g**	
Each 125 g provides	
Energy	430 kJ
Protein	4.5 g
Carbohydrate	11.1 g
Fat	4.5 g

(b) The label on yoghurt B shows different information.

A boy eats the same amount of yoghurt A and yoghurt B.

Which yoghurt provides him with more carbohydrate?

Show your working.

Yoghurt B **150 g**	
Each 150 g provides	
Energy	339 kJ
Protein	6.6 g
Carbohydrate	13.1 g
Fat	0.2 g

From 2001 Key Stage 3 Paper 2 question 11

Translate the data from each question into a useful form.

Part (a)

Yoghurt A	125 g	1 g	25 g	100 g
Protein	4.5 g	?	?	?

Which of the entries is useful to calculate?

Part (b)

Yoghurt A	125 g	1 g	25 g	100 g	150 g
Carbohydrates	11.1 g	?	?	?	?

Yoghurt B	150 g	1 g	25 g	100 g	150 g
Carbohydrates	13.1 g	?	?	?	?

Which of the entries is useful to calculate?

Problem bank for phase 3

These problems support phase 3 of the multiplicative relationships unit. They have been selected from previous Key Stage 3 test papers. Questions 1–3 are targeted at NC level 5, questions 4–8 at level 6, questions 9 and 10 at level 7 and question 11 at level 8. All questions are taken from the calculator paper. Additional problems can be found on pages 5, 75 and 79 of the Framework's supplement of examples.

1 Paint

You can make different colours of paint by mixing red, blue and yellow in different proportions. For example, you can make green by mixing 1 part blue to 1 part yellow.

(a) To make purple, you mix 3 parts red to 7 parts blue.

How much of each colour do you need to make 20 litres of purple paint? Give your answer in litres.

. . . . . litres of red and . . . . . litres of blue

(b) To make orange, you mix 13 parts yellow to 7 parts red.

How much of each colour do you need to make 10 litres of orange paint? Give your answer in litres.

. . . . . litres of yellow and . . . . . litres of red

From 1998 Key Stage 3 Paper 2 question 10

2 Ratios

(a) Nigel pours 1 carton of apple juice and 3 cartons of orange juice into a big jug.

What is the ratio of apple juice to orange juice in Nigel's jug?

apple juice : orange juice = ... : . . . . .

(b) Lesley pours 1 carton of apple juice and $1\frac{1}{2}$ cartons of orange juice into another big jug.

What is the ratio of apple juice to orange juice in Lesley's jug?

apple juice : orange juice = ... : . . . . .

(c) Tandi pours 1 carton of apple juice and 1 carton of orange juice into another big jug. She wants only half as much apple juice as orange juice in her jug.

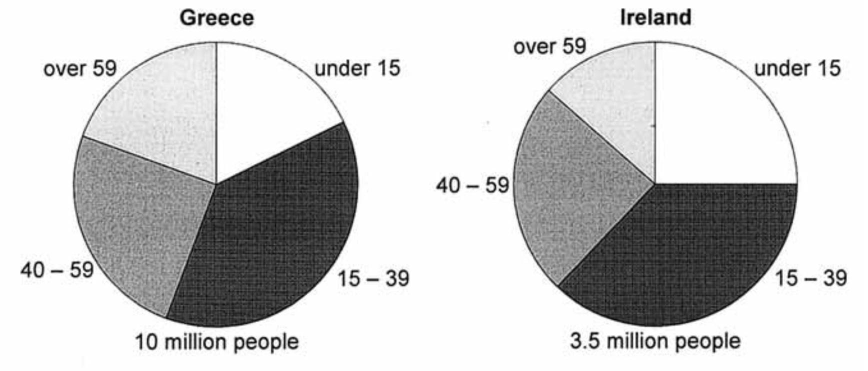
What should Tandi pour into her jug now?

From 1999 Key Stage 3 Paper 2 question 11

3 Ages

These pie charts show some information about the ages of people in Greece and in Ireland.

There are about 10 million people in Greece, and there are about 3.5 million people in Ireland.



(a) Roughly what percentage of people in Greece are aged 40–59?

(b) There are about 10 million people in Greece. Use your percentage from part (a) to work out roughly how many people in Greece are aged 40–59.

(c) Dewi says: ‘The charts show that there are more people under 15 in Ireland than in Greece.’

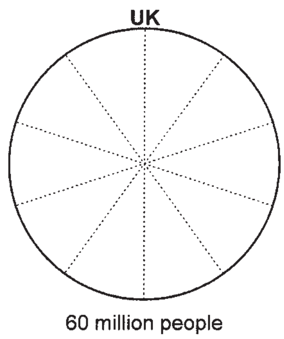
Dewi is wrong. Explain why the charts do not show this.

(d) There are about 60 million people in the UK.

The table shows roughly what percentage of people in the UK are of different ages.

under 15	15–39	40–59	over 59
20%	35%	25%	20%

Draw a pie chart below to show the information in the table. Label each section of your pie chart clearly with the ages.



From 1998 Key Stage 3 Paper 2 question 12

4 Birds

(a) One morning last summer Ravi carried out a survey of the birds in the school garden. He saw 5 pigeons, 20 crows, 25 seagulls and 45 sparrows.

Complete the line below to show the ratios.

1 : . . . . . : . . . . . : . . . . .

(b) What percentage of all the birds Ravi saw were sparrows?

(c) One morning this spring Ravi carried out a second survey. This time he saw:

the same number of pigeons

25% fewer crows

60% more seagulls

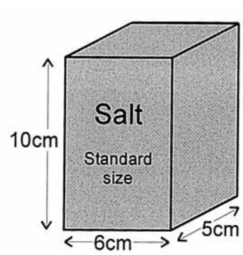
two thirds of the number of sparrows

Pigeons	:	Crows	:	Seagulls	:	Sparrows
1	:	. . . . .	:	. . . . .	:	. . . . .

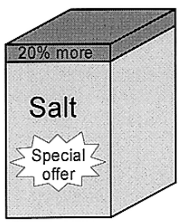
From 1995 Key Stage 3 Paper 2 question 13

5 Salt

(a) What is the volume of this standard size box of salt?



(b) What is the volume of this special offer box of salt, which is 20% bigger?



(c) The standard size box contains enough salt to fill up 10 salt pots.

How many salt pots may be filled up from the special offer box of salt?

From 1996 Key Stage 3 Paper 2 question 12

6 Population

Emlyn is doing a project on world population. He has found some data about the population of the regions of the world in 1950 and 1990.

Regions of the world	Population (in millions) in 1950	Population (in millions) in 1990
Africa	222	642
Asia	1 558	3 402
Europe	393	498
Latin America	166	448
North America	166	276
Oceania	13	26
World	2 518	5 292

(a) In 1950, what percentage of the world's population lived in Asia? Show each step in your working.

(b) In 1990, for every person who lived in North America how many people lived in Asia? Show your working.

(c) For every person who lived in Africa in 1950 how many people lived in Africa in 1990? Show your working.

(d) Emlyn thinks that from 1950 to 1990 the population of Oceania went up by 100%.

Is Emlyn right? Tick the correct box.

☐ Yes ☐ No ☐ Cannot tell

Explain your answer.

From 1996 Key Stage 3 Paper 2 question 10

7 Continents

The table shows the land area of each of the world's continents.

Continent	Land area (in 1 000 km^2)
Africa	30 264
Antarctica	13 209
Asia	44 250
Europe	9 907
North America	24 398
Oceania	8 534
South America	17 793
World	148 355

(a) Which continent is approximately 12% of the world's land area?

(b) What percentage of the world's land area is Antarctica? Show your working.

(c) About 30% of the world's area is land. The rest is water. The amount of land in the world is about 150 million km^2.

Work out the approximate total area (land and water) of the world. Show your working.

From 1998 Key Stage 3 Paper 2 question 2

8 Currency

(a) Use £1 = 9.60 francs to work out how much 45p is in francs. Show your working.

(b) Use 240 pesetas = £1 to work out how much 408 pesetas is in pounds. Show your working.

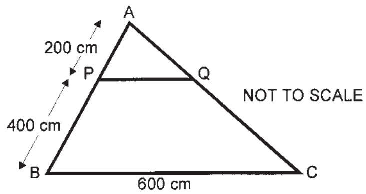
(c) Use £1 = 9.60 francs and £1 = 240 pesetas to work out how much 1 franc is in pesetas. Show your working.

From 1999 Key Stage 3 Paper 2 question 6

9 Roof frames

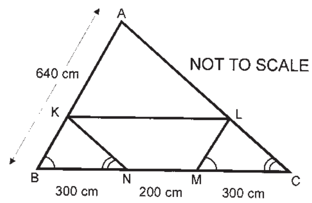
Timpkins Builders make wooden frames for roofs on new houses.

In the diagram of the wooden frame shown below, PQ is parallel to BC.



(a) Calculate length PQ using similar triangles. Show your working.

In the diagram of the wooden frame shown below, angle ABC = angle LMC, and angle ACB = angle KNB.



(b) Calculate length LM using similar triangles. Show your working.

From 1995 Key Stage 3 Paper 2 question 9

10 Pupils

The table shows some information about pupils in a school.

	Left-handed	Right-handed
Girls	32	180
Boys	28	168

There are 408 pupils in the school.

(a) What percentage of the pupils are boys? Show your working.

(b) What is the ratio of left-handed pupils to right-handed pupils?

Write your ratio in the form 1 : . . . . . .

Show your working.

(c) One pupil is chosen at random from the whole school.

What is the probability that the pupil chosen is a girl who is right-handed?

From 1997 Key Stage 3 Paper 2 question 14

11 Births

Look at the table:

	1961	1994
England	17.6	
Wales	17	12.2

(a) In England, from 1961 to 1994, the birth rate fell by 26.1%. What was the birth rate in England in 1994?

Show your working.

(b) In Wales, the birth rate also fell. Calculate the percentage fall from 1961 to 1994.

Show your working.

(c) From 1961 to 1994, the birth rates in Scotland and Northern Ireland fell by the same amount.

The percentage fall in Scotland was greater than the percentage fall in Northern Ireland.

Put a tick by the statement below which is true.

In 1961, the birth rate in Scotland was higher than the birth rate in Northern Ireland. . . . . .

In 1961, the birth rate in Scotland was the same as the birth rate in Northern Ireland. . . . . .

In 1961, the birth rate in Scotland was lower than the birth rate in Northern Ireland. . . . . .

From the information given, you cannot tell whether Scotland or Northern Ireland had the higher birth rate in 1961. . . . . .

From 1998 Key Stage 3 Paper 2 question 9