The National Strategies
Secondary

# Teaching mental mathematics from level 5

## Number

department for
children, schools and families

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# Contents

# Introduction

## What is mental mathematics?

Almost all of mathematics could be described as 'mental' in the sense that engaging in a mathematical task involves thinking. Thus every mathematical problem a pupil tackles must involve several stages of mental mathematics. Pupils actively involved in mental mathematics might be engaged in any combination of:

**interpreting visualising analysing synthesising explaining hypothesising inferring deducing judging justifying making decisions**

These ideas are prevalent throughout mathematics and underpin mathematical processes and applications.

If the definition is so wide ranging, how have we produced a few brief booklets with this title? The answer is that we have been very selective! The 'mental mathematics' supported through the teaching approaches described in these booklets is aimed at a subset of mental mathematics in its broadest sense. We have chosen a few key areas likely to influence pupils' progress beyond level 5. These selections have been informed by recent annual standards reports from the Qualifications and Curriculum Development Agency (QCDA) and the experience of teachers and consultants. The initial ideas have also been supported by classroom trials.

## How do I help pupils to improve the way they process mathematics mentally?

Individual pupils will be at different stages but all pupils develop some strategies for processing mathematical ideas in their heads. Many of the activities suggested in these booklets increase the opportunities for pupils to learn from one another by setting them to work collaboratively on tasks that require them to talk. Often pupils develop and enhance their understanding after they have tried to express their thoughts aloud. It is as if they hear and recognise inconsistencies when they have to verbalise their ideas.

Equally, new connections can be made in a pupil's 'mental map' when, at a crucial thinking point, they hear a different slant on an idea. A more discursive way of working often allows pupils to express a deeper and richer level of understanding of underlying concepts that may otherwise not be available to them. In this way pupils may:

- reach a greater facility level with pre-learned skills, for example, becoming able to solve simple linear equations mentally
- achieve a leap in understanding that helps to complete 'the big picture', for example, seeing how the elements of a function describing the position-to-term relationship in a sequence are generated from elements in the context of the sequence itself.

The activities are designed to engage pupils in group work and mathematical talk.

## Is mental mathematics just about the starter to the lesson?

Developing mental processes is not simply about keeping some skills sharp and automating processes through practice. The activities described in this booklet support the main part of the lesson. Developing a mental map of a mathematical concept helps pupils to begin to see connections and use them to help solve problems. Developing the ability to think clearly in this way takes time. Once in place, some aspects of mental mathematics can be incorporated into the beginning of lessons as a stimulating precursor to developing that topic further.

The activities are intended to support the main part of the lesson.

## Is mental mathematics just about performance in mental tests?

Using these materials will help pupils to perform more successfully in tests, but the aim is more ambitious than that. Developing more effective mental strategies for processing mathematical ideas will impact on pupils' progress in mathematics and their confidence to apply their skills to solve problems.

Secondary teachers recognise the importance of pupils' mathematical thinking and application, but few have a range of strategies to support its development. The expectations described, and the activities suggested in the accompanying mental mathematics resources, aim to create a level of challenge that will take pupils further in their thinking and understanding. These materials should provide the chance for pupils to interact in such a way that they learn from each other's thinking, successes and misconceptions and thereby become increasingly confident and independent learners.

Pupils need to transfer mathematics confidently and apply it whenever they need to use it. This needs to be taught. Most commonly, pupils will use mental mathematics in solving problems as they occur in their lives, in other areas of their studies and as they prepare for the world of work. To support pupils in doing this, teachers will frequently need to set both large and small mathematical problems in real, purposeful and relevant contexts. Pupils will need to solve increasingly complex and unfamiliar problems using mathematics, apply more demanding mathematical procedures during their analysis and do so with increasing independence. These materials support teachers in planning a structured and progressive approach to do this. If learning is planned with mental mathematics as a significant element, pupils will develop increasing confidence in applying mathematics.

Improving mental mathematics will improve pupils' confidence to apply what they know.

## Can mental mathematics involve paper and pencil?

Mathematical thinking involves drawing on our understanding of a particular concept, making connections with related concepts and previous problems and selecting a strategy accordingly. Some of these decisions and the subsequent steps in achieving solutions are committed to paper and some are not. When solving problems, some of the recording becomes part of the final solution and some will be disposable jottings.

Many of the activities involve some recording to stimulate thinking and talking. Where possible, such recording should be made on large sheets of paper or whiteboards. This enables pupils, whether working as a whole class or in pairs or small groups, to share ideas. Such sharing allows them to see how other pupils are interpreting and understanding some of the big mathematical ideas. Other resources such as diagrams, graphs, cards, graphing calculators and ICT software are used in the activities. Many of these are reusable and, once developed in the main part of a lesson, can be used more briefly as a starter on other occasions.

Progress may not appear as written output. Gather evidence during group work by taking notes as you listen in on group discussions. Feed these notes into the plenary and use them in future planning.

# The materials

Each attainment target in mathematics is addressed through its own booklet, divided into separate topic areas. For each topic, there is a progression chart that illustrates expectations for mental processes, broadly from level 5 to level 8. Mathematical ideas and pupils' learning are not simple to describe, nor do they develop in a linear fashion. These are not rigid hierarchies and the degree of demand will be influenced by the context in which they occur and, particularly for the number topics, by the specific numbers involved. For this reason ideas from one chart have to interconnect with those in another. The aim is that the charts will help teachers to adjust the pitch of the activities that are described on subsequent pages.

There are many National Strategies materials which reinforce and extend these ideas but, to ensure that these booklets are straightforward and easy for teachers to use, cross-referencing has been kept to a minimum. The most frequent referencing throughout the booklet is to the *Supplement of examples*, which is now connected to the Framework for Secondary Mathematics (**www.standards.dcsf.gov.uk/nationalstrategies**). The page numbers of the original supplement have been retained and the examples can be downloaded as a complete document or in smaller sets from the related learning objectives.

# Teaching mental mathematics from level 5: Number

The topics covered in this chapter are:

- place value, ordering and rounding
- fractions, decimals, percentages, ratio and proportion.

These are selected from the number (numbers and the number system) section of the learning objectives on the *Framework for secondary mathematics*. The suggested activities target some aspects of number which pupils continue to find difficult. Planning sequences of lessons to include these types of activities will:

- enable pupils to understand at which stage it is appropriate to round numbers and how to do so in different contexts
- reinforce pupils' understanding of fractions, to support them when they calculate with fractions
- develop pupils' understanding of percentages so that they can confidently find percentages of quantities, express one quantity as a percentage of another and increase or decrease a quantity by a percentage
- develop pupils' understanding of the role of multiplication in proportional reasoning.

Pupils still need to use and refine their mental calculation strategies, even when they are using written or calculator methods for working out more complex calculations. A secure understanding of place value underpins calculation and enables pupils to work with both large and small numbers.

The activities described in this chapter address some typical numerical misconceptions. They are designed to engage pupils with number through discussion and collaborative work. The aim is to develop pupils' understanding of the mental mathematics underpinning place value and proportional reasoning. The tasks may easily be adapted to adjust the level of challenge and keep pupils at the edge of their thinking.

In calculation up to level 5, many pupils use a mental image of the number line to support their strategies. Throughout Key Stage 3, as pupils extend their understanding of the number system, it can be helpful to focus more attention on the structure of the base-10 number system as shown on a place-value chart. This provides a good foundation for work beyond level 5, for example, when using standard form.

In the following sections three teaching strategies are used repeatedly and are worth general consideration.

- **Modelling** provides an opportunity for the teacher to make explicit the skills, processes and links that would otherwise be hidden from pupils or unclear to them. Teachers share their thinking so that mental processes are made explicit. Pupils become increasingly involved as they are encouraged to think about the task, ask questions, offer contributions and test ideas. The oral rehearsal of ideas also provides pupils with a good model, which they can then develop in small groups. Ensure that this activity is more than 'teacher talk'. Provide pupils with resources to match the display model and encourage them to give commentaries of their own. Design tasks for pupils to tackle, after the teacher's modelling session, that will make talk essential.
- **Classifying** is a task well suited to the thinking processes that everyone uses naturally to organise information and ideas. A typical classification task may involve a card sort. Pupils work together to sort cards into groups with common characteristics that establish criteria for classification.

Being asked to consider and justify their criteria helps pupils to develop their skills and understanding. The key part of designing a good classification task is the initial choice of cards that will provide a sufficiently high challenge. A common mistake when running a classification task is to intervene too soon and over-direct the pupils.

- **Matching** different forms of representation often involves carefully-selected cards and a common lesson design. In this instance pupils are asked to match cards that are equivalent in some way. This kind of activity can give pupils important mental images, at the same time offering the chance to confront misconceptions.

# Place value and ordering

In primary school pupils will have worked extensively with the image of the number line. As their knowledge and experience of mathematics extends they will welcome the place-value chart as a most flexible and useful image. It will help extend their ability to deal with large and small numbers.

The ability to multiply and divide by any integer power of 10 and to start writing numbers in standard form depends on a secure understanding of place value. This understanding is fundamental in manipulating large and small numbers, both mentally and in written form.

## Multiply and divide numbers by powers of 10

Use positive integer powers of 10 and refer to prior knowledge of the way in which a division fact can be derived from a known multiplication fact. Include the vocabulary of multiplication and division as inverse operations.

| | |
|---|---|
| $5.32 \times 10 = 53.2$<br>$53.2 \div 10 = 5.32$<br>$6.95 \times 100 = 695$<br>$695 \div 100 = 6.95$<br>$4.78 \times 1000 = 4780$<br>$4780 \div 1000 = 4.78$ | Begin with 10, 100, etc. |
| $6 \times 0.1 = 0.6$<br>$0.6 \div 0.1 = 6$<br>$7 \times 0.01 = 0.07$<br>$0.07 \div 0.01 = 7$ | Extend the same approach and understanding to multiplying and dividing by 0.1, 0.01, etc. |

## Understand the effect of multiplying and dividing by numbers between 0 and 1

Use powers of 10 as the multiplier to help pupils recognise the logic of the emerging pattern.

| | | | | |
|---|---|---|---|---|
| $4.2 \times 100$ | = | 420 | Makes bigger | Understand that multiplying by any number between 0 and 1 makes the number smaller |
| $4.2 \times 10$ | = | 42 | ↑ | |
| $4.2 \times 1$ | = | 4.2 | **Stays the same** | |
| $4.2 \times 0.1$ | = | 0.42 | ↓ | |
| $4.2 \times 0.01$ | = | 0.042 | Makes smaller | |
| $4.2 \div 0.01$ | = | 420 | Makes bigger | Understand that dividing by any number between 0 and 1 makes the number bigger |
| $4.2 \div 0.1$ | = | 42 | ↑ | |
| $4.2 \div 1$ | = | 4.2 | **Stays the same** | |
| $4.2 \div 10$ | = | 0.42 | ↓ | |
| $4.2 \div 100$ | = | 0.042 | Makes smaller | |

## Multiply and divide decimals by any number between 0 and 1

Use mental calculations with whole numbers and adjust, using knowledge of the effect of multiplying or dividing by numbers between 0 and 1.

| **Multiplying** | |
|---|---|
| $31 \times 0.4$ | $31 \times 4 = 124$<br>10 times smaller is 12.4 |
| $0.25 \times 0.03$ | $0.25 \times 3 = 0.75$<br>100 times smaller is 0.0075 |

| **Dividing** | |
|---|---|
| $81 \div 0.3$ | $81 \div 3 = 27$<br>10 times bigger is 270 |
| $0.24 \div 0.06$ | $0.24 \div 6 = 0.04$<br>100 times bigger is 4 |

Alternatively, use the definition of fractions as division and knowledge of equivalent fractions.

| **Dividing** | |
|---|---|
| $81 \div 0.3 = 27$ | $\frac{81}{0.3} = \frac{810}{3} = \frac{270}{1} = 270$ |
| $0.24 \div 0.06$ | $\frac{0.24}{0.06} = \frac{24}{6} = \frac{4}{1} = 4$ |

## Begin to write numbers in standard form

Use movements on a place-value grid. Relate numbers back to the 'baseline' of unit digits and describe movements in terms of multiplication, first by multiples of 10 then by powers of 10.

| | |
|---|---|
| $235.7 = 2.357 \times 10^2$ | Write large numbers in standard form |
| $0.000\,92 = 9.2 \times 10^{-4}$ | Write small numbers in standard form |
| $6.92 \times 10^{-4}$, $2.5 \times 10^2$, $3.7 \times 10^2$ | Order numbers in standard form |

The *Framework for secondary mathematics* supplement of examples, pages 38 and 39, provides contexts in which pupils should develop mental processes in place value.

## Activities

**Modelling** different ways in which pupils can use a place-value chart is fundamental to the follow-up tasks described below.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1000 | 2000 | 3000 | 4000 | 5000 | 6000 | 7000 | 8000 | 9000 |
| 100 | 200 | 300 | 400 | 500 | 600 | 700 | 800 | 900 |
| 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |

Illustrate the multiplicative relationships within each column of the chart by discussing the multiplication that takes you from one column entry to another, for example:

$0.08 \times 10 = 0.8$, a step up one row

$0.008 \times 100 = 0.8$, a step up two rows.

Link the steps to the powers of ten and establish that multiplying by $10^3$ is the same as multiplying by 1000 and is equivalent to a step up three rows.

Confirm that moving down within a column can be considered as division or multiplication, for example:

- $8 \div 100 = 0.08$, a step down two rows
- $8 \times 0.01 = 0.08$, a step down two rows
- $8 \times 10^{-2} = 0.08$, a step down two rows.

**Hide and reveal** tasks engage pupils in the same movements around the chart. Use blank strips to **hide** rows of numbers and ask pupils to explain how they know what numbers are under the strip. Alternatively, **reveal** only one row of the chart and ask pupils to complete the rows above and below it. As a more challenging task ask for the row three steps above or below.

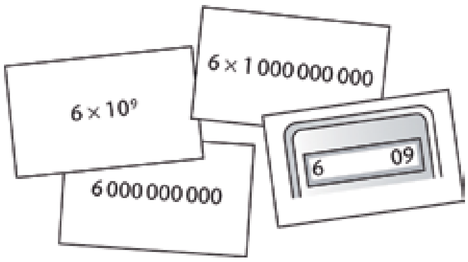
Windows in the chart can be used as the basis of a task in which pupils are asked to create the surrounding entries or even the whole chart, possibly using a spreadsheet.

| | |
|---|---|
| 0.05 | 0.06 |
| 0.005 | 0.006 |

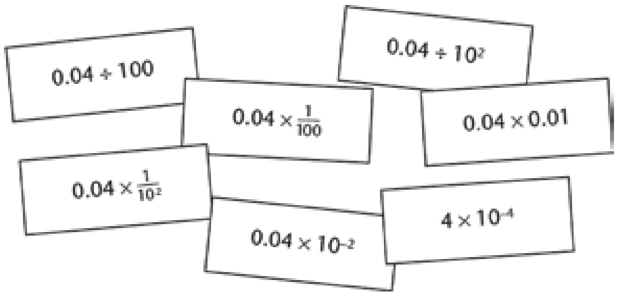
This chart can reinforce pupils' understanding of the effect of multiplying by numbers smaller than 1. It also allows discussion of the fact that the steps can be reversed in two ways:

- by using the same operation (multiplication) with an inverse operator $10^{-4}$ or 0.0001
- by using the same number ($10^4$ or 10 000) and an inverse operation, division.

**Matching** different forms of representation can provide pupils with the chance to confront misconceptions. For example, pupils could be asked to match cards that show numbers with cards that show calculations and calculator displays in standard form. For example:



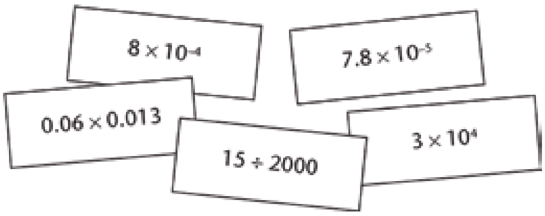

Alternatively, the cards might only show the same calculation written in different ways.




Note that in the task above, and in the one below, numbers are written in standard form without necessarily using the term 'standard form'. The introduction to standard form itself is described in the next modelling activity.

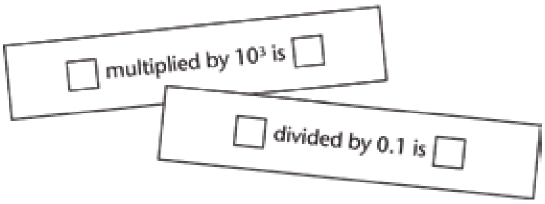
**Ordering** without using a calculator focuses attention on place value. Write multiplication and division calculations (including numbers in standard form) on cards and ask pupils which cards they can put in sequence, in ascending or descending order, **without calculating the values of the numbers.**



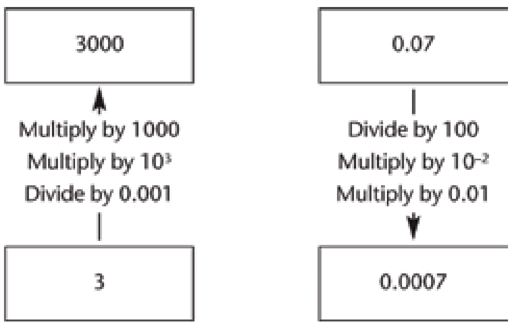

Using cards to match and sequence encourages pupils to discuss their calculations and justify their solutions. Using different sets of cards for different groups of pupils provides a straightforward way of differentiating.

**Fill in the missing numbers** is an activity in which pupils write in the numbers on cards such as those shown below.



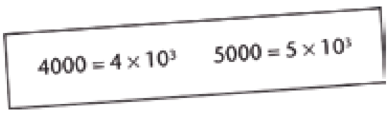

Pupils work in pairs, using a collection of cards with blank entries. They refer to A4 copies of a place-value chart to support the task. This is about using the structure of the chart, **not** about calculating.

**Fill in the missing operations** is the reverse of the above task. Pupils continue to work in pairs and to use the structure of the A4 place-value chart. They should describe the operation needed to get from the start number to the second number in as many ways as they can. The task is simplified if only multiplication is used.




The aim is for pupils to become confident in making multiplicative statements linking pairs of numbers from the same column. Their increased confidence will be evidence of a greater understanding of the nature of the base-10 number system.

**Modelling** how to write numbers in standard form can also begin with the place-value chart. Identify the 'units' digit line as the baseline of the whole chart. All other lines can be generated from this one by using multiplication to step up or down from it. Ask pupils to describe the steps.



Generalise to show that all the numbers in the 'thousands' row will be represented by a units digit multiplied by $10^3$ ($n \times 10^3$). Give pupils copies of one column from the chart and ask them to annotate each of the entries.

| | |
|---|---|
| $\times 10^5$ | 500 000 |
| $\times 10^4$ | 50 000 |
| $\times 10^3$ | 5000 |
| $\times 10^2$ | 500 |
| $\times 10^1$ | 50 |
| $\times 10^0$ | 5 |
| $\times 10^{-1}$ | 0.5 |
| $\times 10^{-2}$ | 0.05 |
| $\times 10^{-3}$ | 0.005 |
| $\times 10^{-4}$ | 0.0005 |

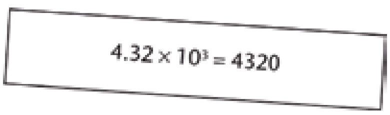
Encourage pupils to speculate how to write 4500 in this way. Referring to the place-value chart, ask:

- Where would you start from?
- Where would you place the resulting number?

Encourage pupils to imagine a number line running across each row. Explain:

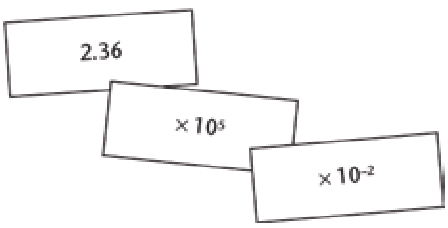
- This helps you to think about 4.5 and 4500 relative to the numbers shown on the chart.

For example, establish that 4.32 is between 4 and 5, closer to 4 than 5. Ask pupils to perform the same operation for 4.32 as above, deciding which multiplication will take it into the row for thousands.



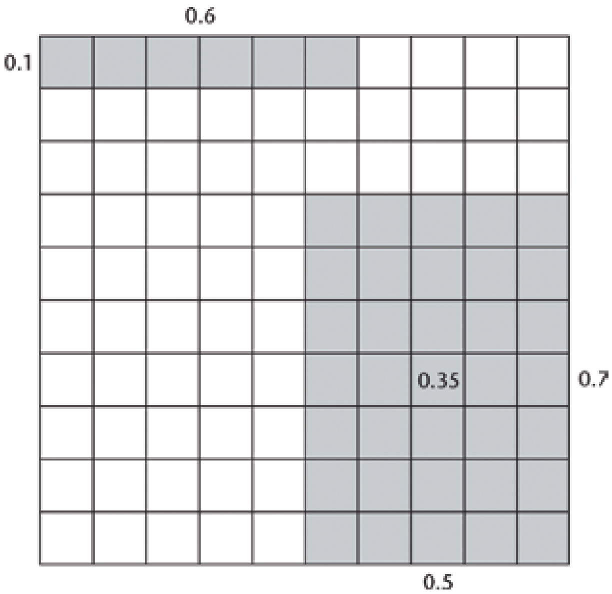

- What is the equation and where would the number lie on the place-value chart?

Show pupils cards with numbers between 1 and 10 and with $\times$ 10 to a suitable index.
Ask them to use the place-value chart to identify the resulting number.



Use a **Diagrammatic explanation** such as a rectangular array to show multiplication and division. Establish first that the unit lengths are divided into tenths and the unit area is divided into hundredths. For example:

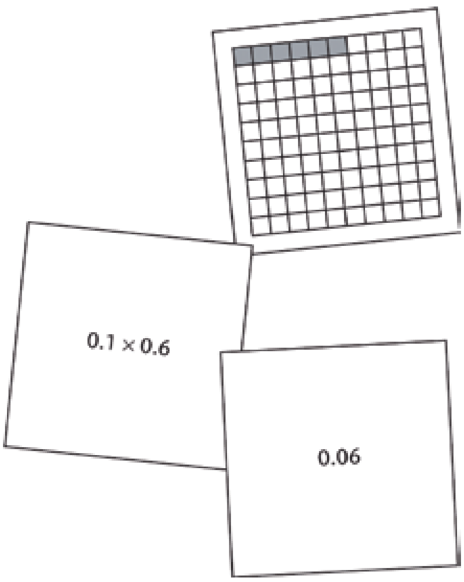
$0.6 \times 0.1 = 0.06$ or $0.35 \div 0.7 = 0.5$




Ask pupils to match arrays and calculations and then to sketch their own arrays for simple decimal multiplication and division calculations.

Challenge pupils to explain how they could extend this array for larger and smaller numbers.

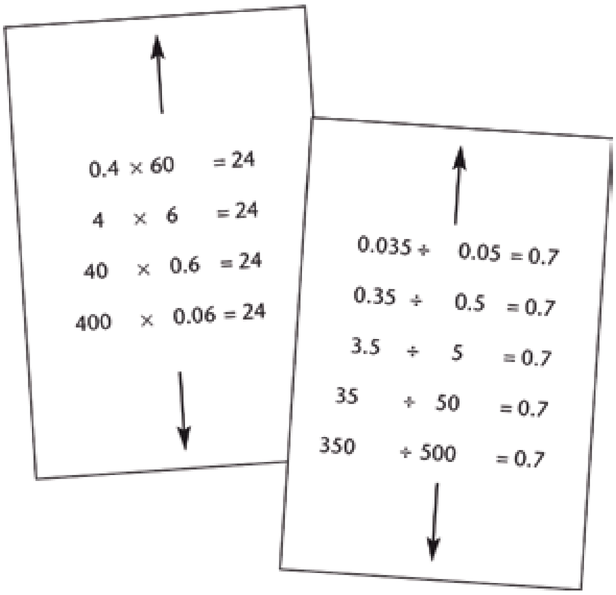
**Matching** a calculation with an answer and a grid representation will consolidate pupils' understanding of this image. For example, ask pupils to match a set from a mixture of cards including examples such as:




The area model is a useful starting point from which to develop understanding of multiplying and dividing by numbers smaller than 1, especially to show how a number can be partitioned into tenths. It provides a powerful visual image for pupils and can aid mental calculation.

This idea is also illustrated in the *Framework for secondary mathematics* supplement of examples, page 39.

Asking pupils to **Find as many ways as possible** helps them to appreciate that a calculation can be written in several ways. It also encourages them to demonstrate that one calculation is equivalent to another. Pupils can also discuss which transformation is the most efficient for each calculation.




Give pupils a calculation and ask them to extend and develop it to find equivalent calculations.

Start from a simple known fact, for example, $4 \times 6 = 24$. Ask questions to lead pupils to consider the effect of making the first number 100 times smaller while the answer remains the same.

- What effect does this have on the magnitude of the second number in the calculation?

Encourage pupils to make generalisations.

Spend some time exploring the different equivalent calculations that are obtained for multiplication and division.

Although pupils may use a calculator to check their answers, they should justify the equivalence through the logic of the calculation.

# Rounding

Mathematics is sometimes regarded as a subject in which all answers must be either right or wrong but there are many occasions in everyday life when an approximate answer is appropriate. For example, the newspaper headline: '75 000 fans watch Manchester United beat Arsenal' does not mean that exactly 75 000 people attended the match. One of two things may have happened. Either people entering the grounds were counted as they passed through the gate, and the exact number was rounded to the nearest 1000 or 5000, or someone estimated the number. Either case is good enough for the newspaper headline.

Similarly, the value of π is often taken as 3.14 because this is sufficiently accurate for the purpose, even though π, an irrational number, cannot be written precisely as a decimal.

In calculations, it is often difficult for pupils who have been accustomed to giving 'right answers' to understand that it is not always either possible or necessary to be exact. They need time to develop the idea of a number being 'suitable for the purpose'.

Estimating an answer before starting a calculation is important as it can reveal subsequent errors, particularly when a calculator is used. So, for example, before they start to multiply $3.7 \times 0.83$, it is useful for pupils to recognise that the answer will be less than 3.7 because they are multiplying by a number less than 1. Rounding the numbers to $4 \times 0.8$ gives 3.2, which is a sufficiently close approximation and can be calculated mentally. The accurate calculated answer, 3.071, may be rounded to an appropriate degree of accuracy.

When solving any problem, pupils need to know the stages at which it is appropriate to round the numbers and the effect this will have on the result.

## Round to whole numbers and specified numbers of decimal places

Work the problem both ways, considering the values that a number might have taken before rounding, i.e. 'unrounding'. Establish criteria for rounding and when a 'trailing zero' is required.

| | |
|---|---|
| 4.48 = 4 (to nearest whole number)<br>4.48 = 4.5 (to 1 decimal place)<br>4.97 = 5 (to nearest whole number)<br>4.97 = 5.0 (to 1 decimal place) | Round decimals to the nearest whole number or to one decimal place |
| 7.499 = 7 (to nearest whole number)<br>7.499 = 7.5 (to 1 decimal place)<br>7.499 = 7.50 (to 2 decimal places) | Round decimals to the nearest whole number or to one or two decimal places |

## Round to specified numbers of significant figures

This way of rounding is most helpful when dealing with very large and very small numbers. It is a precursor to standard form. For large numbers the result can look the same as 'to the nearest 10 000', for example. For very small numbers it allows rounding without everything shrinking to 0. Draw attention to the roles of zero:

- 'leading zeros' as in 0.000 003 76
- 'trailing zeros' as in 458 000
- 'significant zeros' as in 670 006.

In all cases zero is a place-holder maintaining the value of the other digits and therefore the size of the number. In the first two cases the zeros can be established by multiplying or dividing by a power of 10 (as in standard form) but significant zeros cannot be repositioned in this way.

| | |
|---|---|
| 3768 = 4000 (to 1 significant figure)<br>3768 = 3800 (to 2 significant figures)<br>3768 = 3770 (to 3 significant figures)<br>0.002 61 = 0.003 (to 1 significant figure)<br>0.002 61 = 0.0026 (to 2 significant figures)<br>0.0296 = 0.030 (to 2 significant figures)<br>0.005 04 = 0.0050 (to 2 significant figures)<br>2 083 452 = 2 100 000 (to 2 significant figures) | Round decimals to a given number of significant figures |

## Understand upper and lower bounds for discrete and continuous data

Draw attention to the fact that rounding effectively maps an interval of numbers onto a single value.

| | |
|---|---|
| The population of London = 9 million people (to the nearest million)<br>Then the population must be at least 8 500 000 and at most 9 499 999.<br>8 500 000 ≤ (population of London) ≤ 9 499 999 | discrete |
| The distance from Exeter to Plymouth = 62 km (to the nearest km)<br>Then the distance is 61.5 km or further but not as far as 62.5 km.<br>61.5 km ≤ (Exeter to Plymouth) < 62.5 km | continuous |

The *Framework for secondary mathematics* supplement of examples, pages 42 to 47, provides contexts in which pupils should develop mental processes in place value.

## Activities

**Four in a row** gives pupils practice in rounding to one decimal place. Pupils play the game in groups, using counters in two colours, two identical sets of cards numbered 6, 7, 9, 11, 13 and 14 and a 5 × 5 grid. They write 25 numbers, all in the range 0.1 to 1.9 and with one place of decimals, randomly on one 5 × 5 grid, as a baseboard. Pupils take turns to choose two cards (one from each set) and divide one number by the other, using a calculator as appropriate. They round the answer to one decimal place and then place a counter of their own colour on that number on the baseboard. The aim of the game is to get four counters in a row.

This game provides basic practice in rounding to one decimal place and is a good precursor to further activities.

**Before or after?** is a task that involves pupils in considering the different effects of rounding numbers before or after a calculation. Pupils work in pairs, using a pile of cards showing four-digit numbers with three decimal places, such as 4.652, 3.894, 2.453, 8.264, 0.675 and 7.329. They each choose a card and put them together to form a multiplication calculation, for example:

$\boxed{4.652} \times \boxed{3.894}$

First they predict the effect of rounding **before** the calculation. Ask:

- Will the result be smaller or larger than the result of rounding **after** the calculation?
- Is it possible to say?

Then they test their predictions by rounding each number to two decimal places and multiplying them together (with a calculator). They round the result to two decimal places. For example:

4.65 × 3.89 = 18.09

Finally, they multiply the original numbers and round the result to two decimal places.

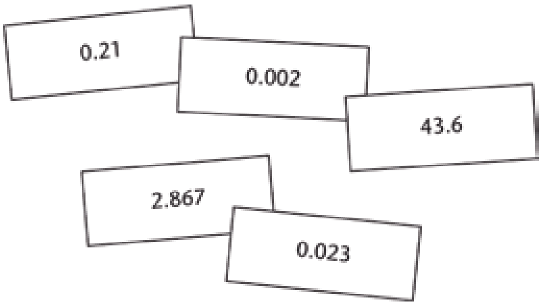
4.652 × 3.894 = 18.11

Encourage pupils to identify and discuss why rounding before computation gives a different answer. This task can be adapted to use different operations. Extend the activity with questions such as:

- Which of the numbers you used gave the greatest difference? Why?
- Which gave the smallest difference? Why?

**Sorting** sets of numbers enables pupils to identify differences and similarities between them. Give pupils some cards showing numbers that could have been rounded to one, two or three decimal places or written correct to one, two or three significant figures.




Ask the pupils to sort these cards into the correct place on a two-way grid.

| Number of significant figures | Number of decimal places: 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | | | | |
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |

Ask questions such as:

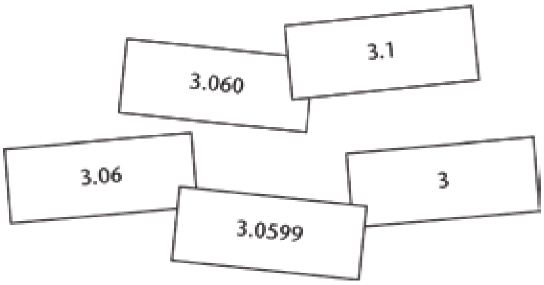
- What numbers can you find that could be placed in empty cells?
- Are there any cells in the table that cannot have an entry?

Once cards have been positioned, the pupils should think about the possible values of each number before rounding. They could then speculate about the upper and lower bounds of each number.

**Same calculation – different problem** is a task in which pupils work out one division calculation that arises from a variety of word problems. The solution for each problem involves the same division calculation but needs a different form of answer or a different degree of accuracy. Some examples are provided on page 24.

**Matching** numbers to the values they might have taken before rounding forces pupils to recognise that a rounded number could represent a range of values.

From a set of prepared cards, pupils should choose a collection that could represent rounded versions of the same number. In each case they should list the numbers in the order implied by the rounded values and state the degree of accuracy to which the number on the card could have been rounded. Finally, based on the evidence they have, they could add the upper and lower bounds of the collection. For example:




These numbers could be listed as 3 (to 1 s.f.), 3.0599 (to 4 d.p.), 3.06 (to 2 d.p.), 3.060 (to 4 s.f.), 3.1 (to 2 s.f.), with lower bound 3.059 85 and upper bound 3.059 95.

Ensure that pupils are given sets of cards that will provide some conflict, including some cards that do not belong to any collections.

Extend the activity by adding another card to a collection and asking:

- What changes do you now need to make to the lower and upper bounds?

As a further extension, ask pupils to make up some cards of their own to challenge another group.

# Fractions, decimals, percentages, ratio and proportion

After calculation, the application of proportional reasoning is the most important aspect of elementary number work. Proportionality underlies key aspects of number, algebra, shape, space and measures, and handling data.

In order to make progress through levels 6, 7 and 8 in the number section of the National Curriculum, pupils must recognise which number to consider as 100% or a whole. This enables them to reverse proportional change and to calculate repeated proportional change. In many problems, making the appropriate choice between fractions, decimals, percentages or ratio will be crucial. Pupils can only make such a choice if they have a sound understanding of equivalence.

Developing mental mathematics from level 5, in this context, means securing a flexible approach to a problem, underpinned by confidence with equivalence and multiplicative strategies.

The following pages in this section focus on:

- understanding and using equivalencies between fractions, decimals and percentages
- using proportional reasoning to solve a problem.

## Understanding and using the equivalencies between fractions, decimals and percentages

Pupils should already have some knowledge of the equivalences between different representations of fractions, decimals and percentages. As they progress beyond level 5 it is important they recognise that the choice of form can affect the ease and efficiency with which a calculation can be performed, particularly mentally. They need to become confident in spotting appropriate forms and converting between them. This is usually a mental process, even when the subsequent calculation is performed on paper or by calculator.

Build this confidence and fluency by creating chains of equivalences, using striking visual arrangements to draw attention to the structures and patterns.

| **Know and use the equivalence between fractions, decimals, percentage and ratio:** | |
|---|---|
| Know that $\frac{3}{5}$, 3/5 and $3 \div 5$ all mean the same | Relate fractions to division |
| Know that $7 \times \frac{1}{3}$, $7 \div 3$, $\frac{1}{3}$ of 7, $\frac{7}{3}$ and $2\frac{1}{3}$ | Interpret different meanings of fractions are equivalent |
| $0.365 = \frac{365}{1000} = \frac{73}{200}$<br>$0.365 = 36.5\%$ | Convert terminating decimals to fractions or percentages |
| $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{11}$ | Convert common recurring decimals to fractions |
| $\frac{1}{3} = 0.3333\ldots = 0.\dot{3}$<br>$\frac{2}{9} = 0.2222 = 0.\dot{2}$<br>$\frac{7}{6} = 1.1666\ldots = 1.1\dot{6}$ | For simple fractions with recurring decimal equivalents, convert between the fraction and decimal forms |
| $\frac{1}{3}, \frac{a}{3a}, \frac{2(a+b)}{6(a+b)}$ | Simplify or find equivalent algebraic fractions |

## Activities

The *Framework for secondary mathematics* supplement of examples, pages 3 to 5, 60 to 81 and 98 to 101, provides contexts in which pupils should develop mental processes in fractions, decimals, percentages, ratio and proportion.

**Within and between** is a task in which pupils use arrays to simplify or find equivalent fractions or ratios. For example:

| | Equivalent fractions | | |
|---|---|---|---|
| Numerator | 0.7 | 7 | 35 |
| Denominator | ? | ? | 150 |

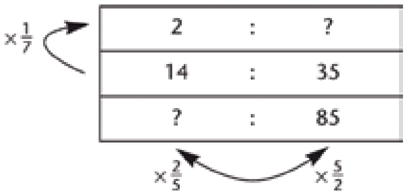
$\times \frac{30}{7}$

$\times \frac{1}{5}$

Pupils can simplify the single multipliers as shown; for example, to get from 7 to 35, use $\frac{35}{7}$ or 5, to get back to 7 from 35 use $\frac{7}{35}$ *or* $\frac{1}{5}$. They can write the calculation, $150 \times \frac{1}{5}$, to find the number, 30, to write in the cell. They should notice that the link between the numerator and denominator is the same for every pair. The link between a pair of numerators (and their equivalent denominators) is different for different pairs.

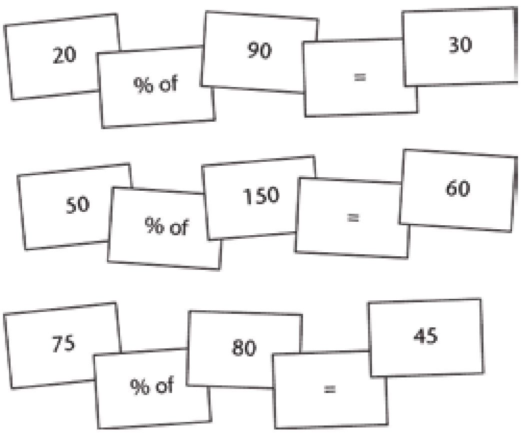
When working with ratio the array would look similar.




Observation and explanation of this 'within and between' relationship for a proportional set is very important. The emphasis is on noting down the calculations, for example, $150 \times \frac{30}{7}$, rather than the actual computation.

**Sorting** activities in the form of a puzzle can help pupils to find different equivalences.

Pupils work in pairs to sort a set of number cards and arrange them to make three correct calculations.



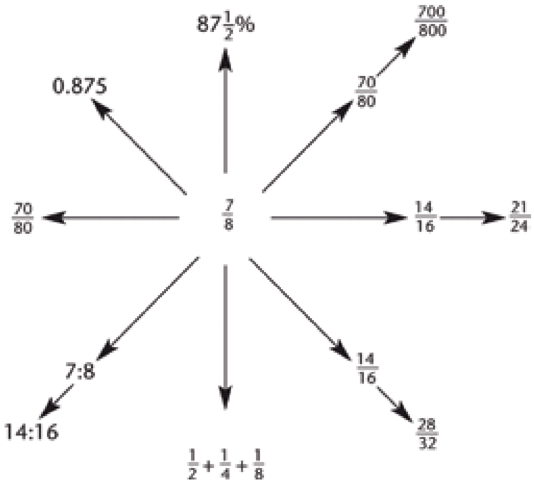

This activity requires pupils to calculate mentally while sorting and resorting the cards until they make all the calculations correct. The activity may be varied by using fractions or decimals rather than percentages.

**Percentages: a chain of reasoning** is a task that encourages pupils to think flexibly about a calculation. It also encourages pupils to give reasons and justify steps they are taking. In the example on page 24 the chain ultimately shows that 60% of 25 is the same as 25% of 60.

**Clouding the picture** is a technique to enable pupils to identify other related facts by using equivalence. For example, ask pupils to complicate the central fraction $\frac{7}{8}$ in *as many ways as they can*. They should:

- start by giving another couple of examples along each branch
- stop after a few examples and try to explain what is happening along the branch *(generalising the process)*
- start a new branch that does something different to complicate the fraction.




This technique could be modelled by the teacher and then repeated by pupils, using different fractions, decimals, percentages or ratios as the centre numbers. The task can easily be adjusted to match the challenge to the group of pupils. Another example is shown on page 25.

## Using proportional reasoning to solve a problem

Develop the concept of a single multiplier, using diagrams or tabular arrays to represent the data.

| | |
|---|---|
| 0.625 is equivalent to $\frac{5}{8}$ and 62.5% | Express a decimal as a fraction or a percentage |
| 21% of £3000 | Interpret percentage as the operator 'so many hundredths of' |
| $\frac{2}{3}$ of 75 | Interpret a fraction as the operator |
| 2 as a percentage of 8 is 25% | Express one number as a percentage of another |
| 5 as a fraction of 3 is $\frac{5}{3}$ or $1\frac{2}{3}$ | Express one number as a fraction of another |
| An increase of 23% is equivalent to a single multiplier of 1.23.<br>A decrease of 23% is equivalent to a single multiplier of 0.77. | Find the outcome of a given percentage increase or decrease |
| In a pastry recipe the ratio of fat to flour is 1:2. For every 1 g of fat there are 2 g of flour so the weight of the fat is $\frac{1}{2}$ the weight of the flour. In the total weight, 1 g in every 3 g is fat. The proportion of fat in the total weight is $\frac{1}{3}$. | Understand the relationship between ratio and proportion |
| After a pay rise of 5%, Dave was paid £5.04. | Identify 100% when the increase or decrease has already taken place |
| The price of a CD is reduced by 12% and then by a further 7%.<br>$\frac{3}{4}$ of the class are girls and $\frac{1}{2}$ of the boys have their ties on. | Solve problems involving repeated proportional change |

The *Framework for teaching secondary mathematics* supplement of examples, pages 3 to 5, 60 to 81 and 98 to 101, provides contexts in which pupils should develop mental processes in fractions, decimals, percentages, ratio and proportion.

## Activities

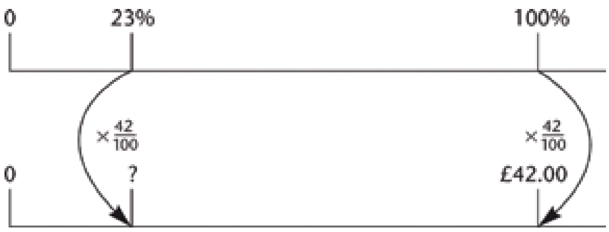
**Aligning diagrams** can help pupils to organise their thinking around the calculations required to solve a problem. They can help to clarify:

- the layout of the data as it is extracted from the problem
- the proportional relationships and hence the layout of the calculations required.

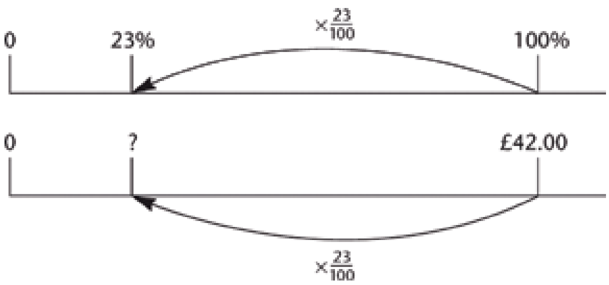
Using a pair of number lines to illustrate multiplicative relationships is a simple alternative to extracting the data into a two-way table. For example:

*Aaron earns £42 a week. He spends 23% of his earnings on CDs. How much money does he spend on CDs each week?*

Show pupils how to extract the relevant information and display it on a number line.




This diagram leads to the calculation $23 \times \frac{42}{1000}$.




This diagram leads to the calculation $42 \times \frac{23}{1000}$.

The key strategy of finding a single multiplier is developed.

The information in the example under **Aligning diagrams** can also be organised into a two-way table.

| | | |
|---|---|---|
| Percentage (%) | 23 | 100 |
| Money (£) | ? | 42.00 |

The problem can be solved by using a single multiplier. Looking at the relationships:

- from right to left gives $42 \times \frac{23}{100}$
- from top to bottom gives $23 \times \frac{42}{100}$

The strength of both these methods is that they need not be changed for more complex problems. For example:

*A shop had a sale. All prices were reduced by 15%. A pair of shoes cost £38.25 in the sale. What price were the shoes before the sale?*

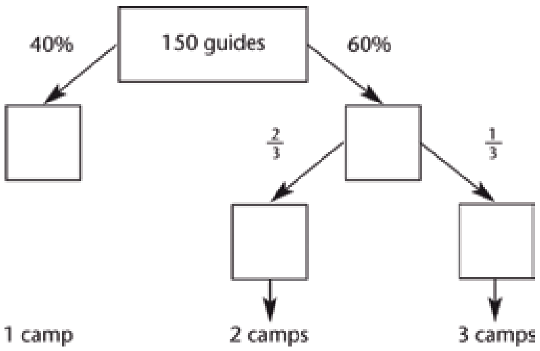

| | | |
|---|---|---|
| Percentage (%) | 85 | 100 |
| Price (£) | 38.25 | ? |

Ask pupils to work in pairs to extract relevant information and then to organise this information on a pair of number lines or in two-way tables for a variety of problems. Ask them to identify the single multipliers and resulting calculations necessary to solve the problem.

These are both good models for solving problems of this type because, when quantities are in proportion, the corresponding ratios within or between these quantities are equal.

A **branching diagram** can model the steps of a problem. For example:

*There are 150 guides at a guide camp. For 40% of them, this is their first camp. Two thirds of the remainder have been to one other camp and the others have all been to three camps in total.*




Pupils could use this type of diagram to organise a variety of problems and then pose additional questions for a partner to solve. This is useful preparation for organising branching diagrams in the context of probability.

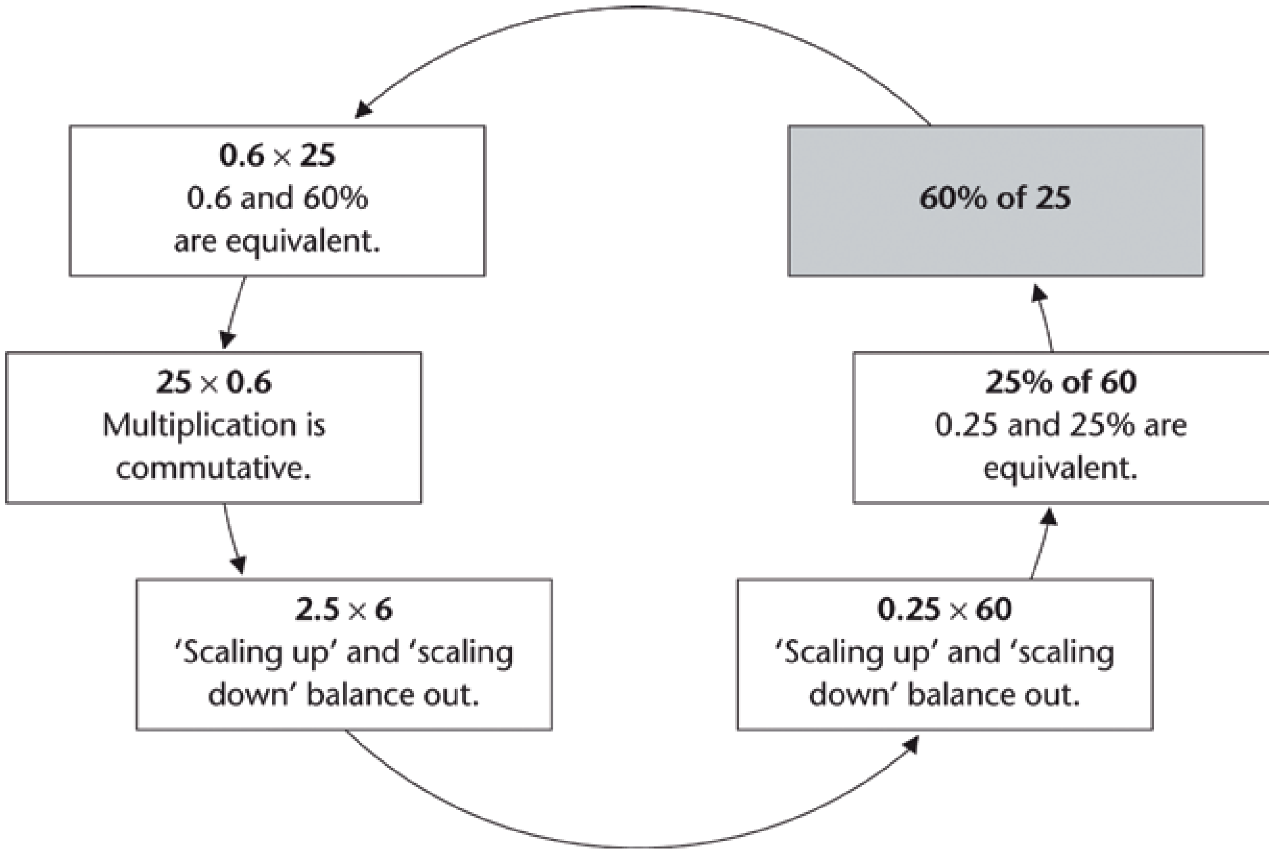
# Examples

## Same calculation – different problem (division problems)

1. The school caretaker knows that 349 people have booked to attend a meeting in the hall. The chairs are to be set out in rows of 18. How many rows are needed?
2. When 18 friends go out for a meal they agree to split the bill of £349 equally. How much should each person pay?
3. Katrina has baked 349 muffins and is packing them in boxes of 18. How many boxes can she fill and how many spare buns will she have?
4. Sam cycles 349 miles in 18 hours. What was his average speed, in miles per hour?
5. Emlyn returned from the USA with $34.90 and exchanged it for pounds at the rate: £1 = $1.80. How much did he get?

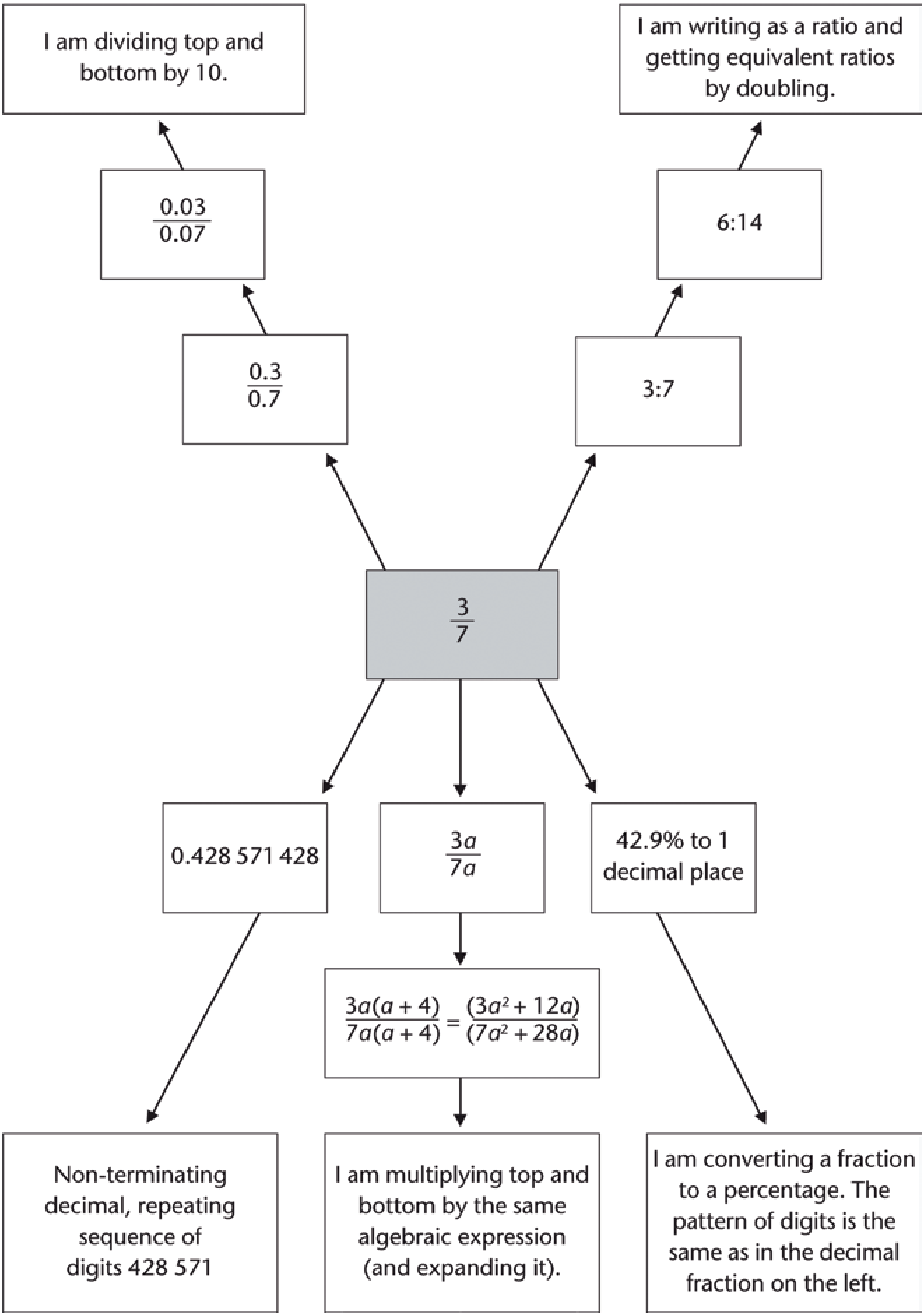
Give another division calculation and ask pupils to make up a new set of problems.

## Percentages: a chain of reasoning



## Clouding the picture: fractions



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