Ratios and equivalence

Rationale

Fractions are first encountered as parts of a whole. At the beginning of Key Stage 3 a broader view is needed because many applications use fractions as a way of comparing one quantity with another. Linking blocks or coloured rods provide a good image, adding a kinaesthetic component to help pupils grasp the relationships. Drawings still have a place, such as when using the OHP or individual pupil whiteboards. The terminology of ‘strips’ and ‘blocks’ was chosen to cover both forms of representation.

Pupils should have knowledge of equivalent fractions from Key Stage 2. However, the approach of building up the number of blocks in each strip establishes equivalences in a way which may be new to them. They will also have some experience of ratio and proportion but may not have met ratio notation before. The strips of blocks provide an effective image for linking ratio and fraction notation.

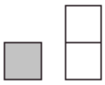
Pupils need to understand the links between different notations. The equivalence between $\frac{1}{2}$ and $1 \div 2$ was discussed in the first two lessons and the link between $1 : 2$ and $\frac{1}{2}$ will be discussed in this lesson. There is an essential equivalence between the expressions, in that they describe the same relationship in different ways: $1 : 2$ stresses the *parts* involved, $1 \div 2$ stresses the *operation* of division and $\frac{1}{2}$ stresses the *result* of the operation.

The main teaching activity and the plenary of this lesson are now described.

Main activity

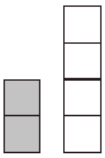
Introducing ratio notation and establishing equivalences

Ask pupils to set up the arrangement on the left, using linking blocks or similar.



We can say that ‘the number of black blocks is $\frac{1}{2}$ the number of white blocks’. Another way of expressing this is ‘the *ratio* of the number of black blocks to the number of white is 1 to 2’, noting that the ratio ‘1 to 2’ is usually written as ‘1 : 2’.

Now ask pupils to double up the number of blocks in each strip, explaining that there is one black block for every two white.

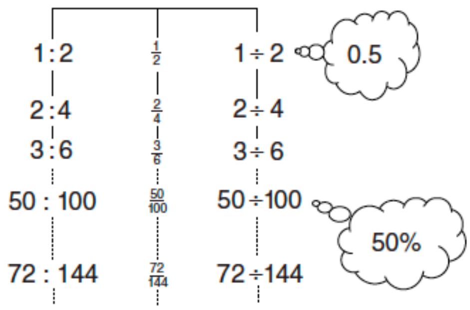


We can say that ‘the number of black blocks is now $\frac{2}{4}$ of the number of white’, or ‘the ratio of the number of black blocks to the number of white is now 2 to 4’. But the number of black is still half the number of white:

‘$\frac{2}{4}$ is equivalent to $\frac{1}{2}$’ and ‘2 : 4 is equivalent to 1 : 2’.

Extend the sequence of equivalent fractions and ratios by adding more blocks to each strip.

Incorporating the link with division from lessons 1 and 2 as well as ratio from this lesson, construct a diagram or table based on $\frac{1}{2}$. Encourage pupils to extend the diagram beyond the equivalences represented with blocks and to add decimals and percentages.



The unit plan suggests repeating the above for other unit fractions, such as $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$. For each, pupils build up a diagram of equivalent ratios, divisions and fractions.

Plenary

Show strips of an unsimplified ratio, such as 6 : 18.

Q What fraction is this equivalent to and why?

Q Could I see this as $\frac{1}{3}$?

Q Could I see this as $\frac{3}{9}$?

Encourage pupils to express ways of 'seeing' the result, rather than talking in terms of 'cancelling' numbers:

- To see $\frac{1}{3}$ pupils may say that three black strips would make the white strip. Or they may say that there is one black block for every three white blocks.
- To explain $\frac{3}{9}$ pupils will need to see the black made up of 3 lots of two and the white made up of 9 lots of two – or 3 black blocks for every 9 white blocks. (Strips suitably coloured may be useful to illustrate.)

Prompts for phase 2

Multiplicative relationships

Rationale

Phase 1 explored fractions as operators and established ratio notation. Phase 2 explores multiplicative relationships within a restricted set of simple fractions/ratios (2 : 5 and 3 : 4 then later 4 : 5 and 2 : 3), expressing operators in fraction, decimal and percentage forms. Following the principle of studying operations and their inverses together sows the seeds of a more formal study later on:

- it encourages flexible thinking, since either of the two quantities involved can be thought of as the unit against which the other is compared;
- it ensures working with operators which are less than 1 and operators which are greater than 1.

Keeping to simple ratios facilitates understanding of the relationships to be expressed. The same principle applies to the sets of problems which follow:

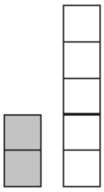
- Pupils can draw on their awareness of the relationships, equivalences and notations they have encountered.
- Their methods may be informal, but they start to think multiplicatively.
- A way of representing the problem is available to them if they wish, or if the teacher thinks it may help them when stuck.

The development of the main part of the first lesson in phase 2 is now described in three stages.

Stage 1

Exploring the ratio 2 : 5

Model the recording for pupils on the board, on the OHP (OHT of resource FR5) or using the ITP *Ratio strips*. Pupils can quickly make their own paper record on a copy of resource sheet FR5.



Ask pupils to use linking blocks of two chosen colours to construct the strips shown on the left.

First express the relationship between the number of blocks in each strip (both ways round) using the language of ratio:

No. of black blocks : no. of white blocks = 2 : 5

No. of white : no. of black = 5 : 2

Now, taking each strip in turn as the 'unit' (worth 'one'), express the relationship between the number of blocks, considering fraction, decimal and percentage equivalents.

- Clarify which strip is being taken as the unit (note: 100% = 1).
- Express each block as a fractional part of the unit.
- Express the second strip as a fraction of the chosen strip.

Think of the *white* strip as unit (fraction).

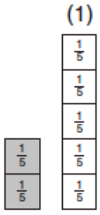
The white strip is worth 'one'.

Q What is each white block worth (as a fraction)?

(Mark each white block as $\frac{1}{5}$.)

Establish each black block is worth $\frac{1}{5}$.

(Mark each black block $\frac{1}{5}$.)



Q What fraction is the number of black blocks of the number of white blocks?

Record:

No. of black blocks = $\frac{2}{5}$ × no. of white blocks

Think of the *black* strip as unit (fraction).

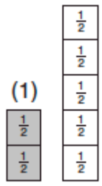
The black strip is worth 'one'.

Q What is each black block worth (as a fraction)?

(Mark each black block as $\frac{1}{2}$.)

Establish each white block is worth $\frac{1}{2}$.

(Mark each white block $\frac{1}{2}$.)



Q What fraction is the number of white blocks of the number of black blocks?

Record:

No. of white blocks = $\frac{5}{2}$ × no. of black blocks

Think of the *white* strip as unit (decimal).

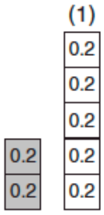
The white strip is worth 'one'.

Q What is each white block worth (as a decimal)?

(Mark each white block as 0.2.)

Establish each black block is worth 0.2.

(Mark each black block 0.2.)



Q What decimal fraction is the number of black blocks of the number of white blocks?

Record:

No. of black blocks = 0.4 × no. of white blocks

Think of the *black* strip as unit (decimal).

The black strip is worth 'one'.

Q What is each black block worth (as a decimal)?

(Mark each black block as 0.5.)

Establish each white block is worth 0.5.

(Mark each white block 0.5.)

Q What fraction is the number of white blocks of the number of black blocks?

Record:

No. of white blocks = 2.5 × no. of black blocks

Think of the *white* strip as unit (percentage).

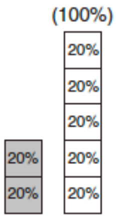
The white strip is worth 100%.

Q What is each white block worth?

(Mark each white block as 20%.)

Establish each black block is worth 20%.

(Mark each black block 20%.)



Q What percentage is the number of black blocks of the number of white blocks?

Record:

No. of black blocks = 40% × no. of white blocks

Think of the *black* strip as unit (percentage).

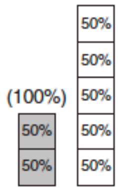
The black strip is worth 100%.

Q What is each black block worth?

(Mark each black block as 50%.)

Establish each white block is worth 50%.

(Mark each white block 50%.)



Q What percentage is the number of white blocks of the number of black blocks?

Record:

No. of white blocks = 250% × no. of black blocks

Points to consider:

- Pupils may want to write $\frac{5}{2}$ as $2\frac{1}{2}$, but do not lose the improper fraction form as a way of expressing the relationship.
- Perhaps introduce the vocabulary of *inverse* operators when talking about the relationships expressed.

Stage 2

Exploring the ratio 3 : 4

Ask the class to work in pairs to complete a similar set of eight expressions for the ratio 3 : 4, using resource sheet FR6 for recording. Follow this with a mini-plenary to check pupils' work and deal with any common difficulties (e.g. it may be necessary to clarify that $\frac{4}{3} = 133\frac{1}{3}\%$).

Stage 3

Solving problems related to the ratios 2 : 5 and 3 : 4

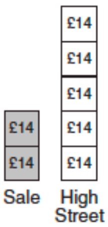
Present a set of problems, all based around the ratios 2 : 5 and 3 : 4 and their inverses. (Resource sheet FR9 gives a varied set of problems which you could adapt to suit your own classes.) Prepare the problems on separate cards or slips of paper, one set per four pupils.

- In groups of four, pupils divide the problems into two sets: those associated with 2 : 5 and those associated with 3 : 4. In some cases, the classification may not be obvious but pupils should be able to sort the problems using informal strategies based on relative sizes.
- Each pair then takes one of the two subsets to solve.
- Pairs check each other's work and discuss problems where the classification was uncertain or incorrect.

It is intended that pupils should use informal methods for solving these problems, drawing on understanding gained from working with the strips, but using the strips for explicit representation of a problem only when it seems helpful. For this reason, it is recommended to let pupils tackle the problems without further guidance, intervening only when they encounter difficulties that they cannot overcome by talking within their pairs or fours.

Here are two examples of how pupils might be helped by using linking blocks and/or sketch diagrams to represent the situation and find, or perhaps just confirm, the answer.

> *An alternative clothing sale offers jeans at £28. These are £70 in high-street shops. Is this less than half price? How could you use fractions, ratios or percentages to compare the alternative price to the high-street price?*

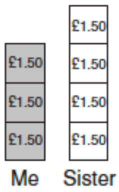


Initial classification of this problem may provoke some discussion. Some pupils might recognise 28 : 70 as equivalent to 2 : 5 and classify accordingly. Or they might argue that it is less than half price and therefore must be related to 2:5 rather than 3:4.

The sale price divides into two parts: $\frac{1}{2} \times £28 = £14$

The high street price divides into five parts: $\frac{1}{5} \times £70 = £14$
(or confirm $5 \times £14 = £70$).

The amounts fit with the ratio 2 : 5. So the sale price is 40% of high-street price.



My sister lets me help on her paper round. It pays £10.50 and she shares this between me and her in the ratio 3 : 4. (a) How much do I get? (b) How much does my sister get?

The diagram shows the £10.50 must be divided into seven parts, each part being $\frac{1}{7} \times$ £10.50 = £1.50.

(a) I get three parts: $\frac{3}{7} \times$ £10.50 = £4.50

(b) My sister gets four parts: $\frac{4}{7} \times$ £10.50 = £6

For the plenary of this lesson, please see the unit plan.

Repeating the lesson with a different data set

Repetition helps to consolidate learning and strengthen pupils' understanding of links. The pattern of the previous lesson can be repeated by changing only the data set:

- choose a different pair of ratios within the set of small numbers 2, 3, 4 and 5, such as 4 : 5 and 2 : 3;
- expect pupils to take a more active role from the beginning, with less need for modelling by the teacher.

Resource sheets FR7 and FR8 are suitable for recording relationships for the two ratios and sheet FR10 gives a set of problems related to these ratios. As before, it is worth cutting the problems up separately, for pupils to sort.

Mixed problems and final plenary

The suggested way of concluding the unit is to give a set of mixed problems for pupils to solve in pairs. Resource sheet FR11 gives a possible set, to be adapted as needed. It includes similar problems to before, but related to mixed ratios/fractions.

For this lesson, plan a mini-plenary to discuss solutions, allowing time for the final full plenary, which refers pupils back to the key lesson and reviews progress made.

In the final plenary, return to the OHTs or poster of ideas from the first lesson. Ask these questions, allowing a short discussion in pairs at each stage:

Q Can we make any additional entries or new links? *(use key words)*

Q What images helped to explain the connections? *(add sketches to the poster)*

We have looked at:

- new ideas and ways of recording these ideas;
- new connections between ideas;
- applying these ideas to solving problems.

Q Which of these has helped you make most progress?

Q What do you need to do more of next time we visit the topic?